A Multiplicity Result for Periodic Solutions of Higher Order Ordinary Differential Equations via the Method of Upper and Lower Solutions*

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We prove a multiplicity result of Ambrosetti–Prodi type problems of higher order. Proofs are based on upper and lower solutions method for higher order periodic boundary value problems and coincidence degree arguments.

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1. INTRODUCTION

This paper is devoted to the study of a multiplicity result for higher order ordinary differential equations of the form

\[ x^{(n)}(t) + f(t, x(t)) = s \quad \text{on } J = [0, 2\pi], \]
\[ x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1, \] (1)

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where $s$ is a real parameter and $f: J \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Throughout this paper, we assume that $f$ is $2\pi$-periodic in the first variable. Assuming the following coerciveness condition

$$\lim_{|x| \to \infty} f(t, x) = \infty \text{ uniformly in } t \in J,$$

we may consider the existence of multiple solutions of (1.5), the so-called Ambrosetti–Prodi type problem. In 1988, among their general set-up of differential operator $L$, Ding and Mawhin [3] have proved under the assumption $f(t, x) = g(x) + e(t, x)$, where $g$ is continuous with the coerciveness condition and $e$ is of Carathéodory type, uniformly bounded by an $L^1$-function, that there exist $s_0$ and $\bar{s}$ with $s_0 \leq \bar{s}$ such that (1.5) has no, at least one or at least two solutions according to $s < s_0$, $s = \bar{s}$ or $s > \bar{s}$. When $n$ is even, they require an additional growth restriction on $g$, i.e. there exists $\gamma \in (0, 1)$ such that

$$(g(x) - g(y))(x - y) \geq -\gamma(x - y)^2, \quad x, y \in \mathbb{R}.$$  

In this case, assuming $e(t, x) = e(t)$ has zero mean value, they also prove that there exists $s_0$ such that (1.5) has no, at least one or at least two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$.

Allowing joint dependence of $(t, x)$ in the nonlinear terms, Ramos and Sanchez [6] deal with a number of situations in which one of the above results can be established. Among others, when $n$ is even and $f$ is continuous and coercive and the following condition holds: there exists $\gamma \in (0, 1)$ such that

$$(f(t, x) - f(t, y))(x - y) \geq -\gamma(x - y)^2, \quad \text{for all } t \in J \text{ and } x, y \in \mathbb{R},$$

they prove the second result in [3].

In this paper, we give a similar result as Ramos and Sanchez [6] with no restriction on the order $n$. More precisely, if $f$ is continuous satisfying (H) and the following condition holds: there exists $M \in (0, A(n))$ such that

$$(f(t, x) - f(t, y))(x - y) \geq -M(x - y)^2, \quad \text{for all } t \in J \text{ and } x, y \in \mathbb{R},$$  

(\text{H1})
where \( A(n) = n!/\pi^n(n-1)^{n-1} \), then \((1_\alpha)\) satisfies the conclusion of the second result in [3].

The proof is based on the method of upper and lower solutions for higher order ordinary differential equations introduced in [2] and an application of coincidence degree.

In what follows, \( J = [0, 2\pi] \). Mean value \( \bar{x} \) of \( x \) and the function \( \bar{x} \) of mean value 0 will be respectively defined by
\[
\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t) \, dt
\]
and
\[
\bar{x}(t) = x(t) - \bar{x}.
\]
\( C^k(J) \) will denote the space of continuous functions defined on \( J \) into \( \mathbb{R} \) whose derivative through order \( k \) are continuous, \( C_{2\pi}^k(J) \) the space of \( 2\pi \)-periodic functions of \( C^k(J) \), \( L^p(J) \) the classical real Lebesgue space with the usual norm \( \| \cdot \|_p \). \( W^{k,1}(J) \) denotes the Sobolev space of all functions \( x \) of \( C^{k-1} \), with \( x^{(k-1)} \) absolutely continuous and \( W^{k,1}_{2\pi}(J) \) the space of \( 2\pi \)-periodic functions of \( W^{k,1} \).

### 2. Maximum Principles and the Method of Upper and Lower Solutions

Let \( \mathcal{L}_n : F^n_{2\pi} \to L^1(J) \) be defined by \( \mathcal{L}_n \equiv D^n + MI \), where \( D = d/dt \), \( I \) is the identity operator, \( M \) is a nonzero real constant, and
\[
F^n_{2\pi} = \{ x \in W^{n,1}(J) : x^{(i)}(0) = x^{(i)}(2\pi), \ i = 0, \ldots, n-2, \ x^{(n-1)}(0) \geq x^{(n-1)}(2\pi) \}.
\]

**Definition 1** We say that \( \mathcal{L}_n \) is inverse positive in \( F^n_{2\pi} \) if \( \mathcal{L}_n x \geq 0 \) implies \( x \geq 0 \), for all \( x \in F^n_{2\pi} \) and \( \mathcal{L}_n \) is inverse negative if \( \mathcal{L}_n x \geq 0 \) implies \( x \leq 0 \), for all \( x \in F^n_{2\pi} \).

We present some maximum principles for the operator \( \mathcal{L}_n \).

**Lemma 1** (Cabada [1]) Let \( A(n) = n!/\pi^n(n-1)^{n-1} \). Then the operator \( \mathcal{L}_n \) is inverse positive in \( F^n_{2\pi} \) for \( M \in (0, A(n)) \), and \( \mathcal{L}_n \) is inverse negative in \( F^n_{2\pi} \) for \( M \in (-A(n), 0) \).

We notice that the second statement of Lemma 1 can be restated as follows; \( D^n - MI \) is inverse negative in \( F^n_{2\pi} \) for \( M \in (0, A(n)) \).

**Remark 1** By Lemmas 2.1 and 2.2 in [1], we have a strict inequality version of Lemma 1 as follows; if \( M \in (0, A(n)) \) \( (M \in (-A(n), 0)) \), then \( \mathcal{L}_n x > 0 \) implies \( x > 0 \) \( (x < 0) \) in \( F^n_{2\pi} \).
Consider the periodic boundary value problem of higher order
\[
 x^{(n)}(t) + f(t, x(t)) = 0 \quad \text{a.e. on } J, \\
 x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1, \tag{2}
\]
where \( f: J \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, i.e. \( f(\cdot, x) \) is measurable for each \( x \in \mathbb{R} \), \( f(t, \cdot) \) is continuous for a.e. \( t \in J \), and for every \( r > 0 \) there exists \( h_r \in L^1(J) \) such that
\[
|f(t, x)| \leq h_r(t),
\]
for a.e. \( t \in J \) and all \( x \in \mathbb{R} \) with \( |x| \leq r \). We define lower and upper solutions of Eq. (2);

**DEFINITION 2** \( \alpha \in W^{n,1}(J) \) is called a lower solution of (2) if
\[
\alpha^{(n)}(t) + f(t, \alpha(t)) \geq 0 \quad \text{a.e. } t \in J, \\
\alpha^{(i)}(0) = \alpha^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 2, \\
\alpha^{(n-1)}(0) \geq \alpha^{(n-1)}(2\pi).
\]
Similarly, \( \beta \in W^{n,1}(J) \) is called an upper solution of (2) if
\[
\beta^{(n)}(t) + f(t, \beta(t)) \leq 0 \quad \text{a.e. } t \in J, \\
\beta^{(i)}(0) = \beta^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 2, \\
\beta^{(n-1)}(0) \leq \beta^{(n-1)}(2\pi).
\]

The following theorem is proved by Cabada [2], but here we give a different proof for reader’s convenience, since part of the proof is useful to continue arguments in the proof of Theorem 3 in Section 3. The proof essentially follows Theorem 1.1 in [4].

**THEOREM 1** Assume that \( \alpha \) and \( \beta \) are lower and upper solutions of (2) respectively with \( \alpha(t) \leq \beta(t) \), for all \( t \in J \). Also assume that \( f \) satisfies that there exists \( M \in (0, A(n)) \) such that
\[
f(t, \alpha(t)) + M\alpha(t) \leq f(t, x) + Mx \leq f(t, \beta(t)) + M\beta(t), \tag{H2}
\]
for a.e. \( t \in J \) with \( \alpha(t) \leq x \leq \beta(t) \). Then (2) has a solution \( x \) such that \( \alpha(t) \leq x(t) \leq \beta(t) \), for all \( t \in J \).
Proof Let us consider the modified problem

\[ x^{(n)}(t) + F(t, x(t)) = 0 \quad \text{a.e. on } J, \]

\[ x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1, \]

where \( F : J \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
F(t, x) = \begin{cases} 
  f(t, \beta(t)) - M(x - \beta(t)), & \text{if } x > \beta(t), \\
  f(t, x), & \text{if } \alpha(t) \leq x \leq \beta(t), \\
  f(t, \alpha(t)) - M(x - \alpha(t)), & \text{if } x < \alpha(t),
\end{cases}
\]

\( M \) is a real constant in \((0, M(n))\). We claim that any solution \( x \) of (3) satisfies \( \alpha(t) \leq x(t) \leq \beta(t) \), for all \( t \in J \) so that it is a solution of (2). Let

\( J_1 = \{ t \in J : x(t) > \beta(t) \} \), \( J_2 = \{ t \in J : \alpha(t) \leq x(t) \leq \beta(t) \} \) and \( J_3 = \{ t \in J : x(t) < \alpha(t) \} \). Then on \( J_1 \),

\[
x^{(n)}(t) - \beta^{(n)}(t) \geq -F(t, x(t)) + f(t, \beta(t)) = -f(t, \beta(t)) + M(x(t) - \beta(t)) + f(t, \beta(t)) = M(x(t) - \beta(t)) \quad \text{a.e.}
\]

On \( J_2 \),

\[
x^{(n)}(t) - \beta^{(n)}(t) \geq -f(t, x(t)) + f(t, \beta(t)) \geq M(x(t) - \beta(t)) \quad \text{a.e. by (H2)}.
\]

On \( J_3 \),

\[
x^{(n)}(t) - \beta^{(n)}(t) \geq -F(t, \alpha(t)) + f(t, \alpha(t)) + f(t, \beta(t)) \geq M(x(t) - \alpha(t)) - M(\beta(t) - \alpha(t)) \quad \text{by (H2)}
\]

\[
= M(x(t) - \beta(t)) \quad \text{a.e.}
\]

Thus by the above three cases, we get

\[
x^{(n)}(t) - \beta^{(n)}(t) - M(x(t) - \beta(t)) \geq 0 \quad \text{a.e. } J.
\]

It is not hard to check that \( x - \beta \in F^{n}_{2\pi} \) and thus by Lemma 1,

\[
x(t) \leq \beta(t) \quad \text{for all } t \in J.
\]
Obviously, a similar argument applies to show that
\[ \alpha(t) \leq x(t) \quad \text{for all } t \in J. \]

Therefore we get
\[ \alpha(t) \leq x(t) \leq \beta(t), \quad \text{for all } t \in J. \]

It remains to prove that (3) has at least one solution. To this purpose, consider the homotopy

\[ x^{(n)}(t) - (1 - \lambda)Mx(t) + \lambda F(t, x(t)) = 0 \quad \text{a.e. on } J \]
\[ x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1, \quad (4) \]

where \( \lambda \in [0, 1] \). First of all, we will obtain \textit{a priori} estimate for all possible solutions of (4). Let \( x \) be a solution of (4). We do the case when \( n \) is odd first. Multiplying both sides of (4) by \( x' \) and integrating on \( J \),

\[ \|x^{(p)}\|_2^2 = (-1)^{(n+1)/2} \lambda \int_J F(t, x(t))x'(t) \, dt, \]

where \( p = (n + 1)/2 \). Now

\[ \int_J F(t, x(t))x'(t) \, dt \]
\[ = \int_{J_1} (f(t, \beta(t)) - M(x(t) - \beta(t)))x'(t) \, dt \]
\[ + \int_{J_2} f(t, x(t))x'(t) \, dt + \int_{J_3} (f(t, \alpha(t)) - M(x(t) - \alpha(t)))x'(t) \, dt \]
\[ = \int_{J_1} (f(t, \beta(t)) + M\beta(t))x'(t) \, dt + \int_{J_2} (f(t, x(t)) + Mx(t))x'(t) \, dt \]
\[ + \int_{J_3} (f(t, \alpha(t)) + M\alpha(t))x'(t) \, dt - M \int_J x(t)x'(t) \, dt. \]

The integral in the last term is 0 and by the Carathéodory condition, integrals \( \int_{J_1} |f(t, \beta(t))||x'(t)| \, dt \), \( \int_{J_2} |f(t, x(t))||x'(t)| \, dt \) and \( \int_{J_3} |f(t, \alpha(t))||x'(t)| \, dt \) are bounded by \( \|h_1\|_1 \|x'\|_\infty \), for \( h_1 \in L^1(J) \) determined by \( \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \) in the definition of Carathéodory
function. Thus we get
\[\|x^{(p)}\|_2^2 \leq \int_J |F(t, x(t))||x'(t)| \, dt\]
\[\leq 3(\bar{M} + \|h_1\|_1)\|x'\|_{\infty}\]
\[\leq \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1)\|x''\|_2\] by Sobolev inequality
\[\leq \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1)\|x^{(p)}\|_2\] by Wirtinger inequality,
where \(\bar{M} = 2\pi M \max\{\|\alpha\|_{\infty}, \|\beta\|_{\infty}\}\). Therefore
\[\|x^{(p)}\|_2 \leq \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1),\]
and by Wirtinger inequality again,
\[\|x'\|_2 \leq \sqrt{\frac{3\pi}{2}}(\bar{M} + \|h_1\|_1).\] (2a)

When \(n\) is even, multiplying both sides of (4) by \(\tilde{x}\) and integrating on \(J\), we get for \(p = n/2\),
\[(-1)^p\|x^{(p)}\|_2^2 = (1 - \lambda)M \int_J x(t)\tilde{x}(t) \, dt - \lambda \int_J F(t, x(t))\tilde{x}(t) \, dt.\]

Now
\[\int_J F(t, x(t))\tilde{x}(t) \, dt\]
\[= \int_{J_1} (f(t, \beta(t)) - M(x(t) - \beta(t)))\tilde{x}(t) \, dt\]
\[+ \int_{J_2} f(t, x(t))\tilde{x}(t) \, dt + \int_{J_3} (f(t, \alpha(t)) - M(x(t) - \alpha(t)))\tilde{x}(t) \, dt\]
\[= \int_{J_1} (f(t, \beta(t) + M\beta(t))\tilde{x}(t) \, dt + \int_{J_2} (f(t, x(t)) + Mx(t))\tilde{x}(t) \, dt\]
\[+ \int_{J_3} (f(t, \alpha(t) + M\alpha(t))\tilde{x}(t) \, dt - M \int_J x(t)\tilde{x}(t) \, dt.\]
Thus

\[-1\|^p\|x^{(p)}\|^2_2 = M \int_J x(t) \ddot{x}(t) \, dt - \lambda \left[ \int_{J_1} (f(t, \beta(t)) + M\beta(t)) \ddot{x}(t) \, dt + \int_{J_2} (f(t, x(t)) + Mx(t)) \ddot{x}(t) \, dt + \int_{J_3} (f(t, \alpha(t)) + M\alpha(t)) \ddot{x}(t) \, dt \right],\]

and we get

\[\|x^{(p)}\|^2_2 \leq M\|\ddot{x}\|^2_2 + 3(M + \|h_1\|_1)\|\ddot{x}\|_\infty \]

\[\leq M\|\ddot{x}\|^2_2 + \sqrt{\frac{3\pi}{2}} (M + \|h_1\|_1)\|x^{(p)}\|_2 \quad \text{by Sobolev inequality}\]

\[\leq M\|x^{(p)}\|^2_2 + \sqrt{\frac{3\pi}{2}} (M + \|h_1\|_1)\|x^{(p)}\|_2 \quad \text{by Wirtinger inequality}.\]

Since \(0 < M < 1\), we get

\[\|x^{(p)}\|_2 \leq \frac{\sqrt{3\pi}(M + \|h_1\|_1)}{\sqrt{2}(1 - M)},\]

and by Wirtinger inequality,

\[\|x'\|_2 \leq \frac{\sqrt{3\pi}(M + \|h_1\|_1)}{\sqrt{2}(1 - M)}. \quad (2b)\]

Therefore both (2a) and (2b) imply that

\[\|x'\|_2 \leq \frac{\sqrt{3\pi}(M + \|h_1\|_1)}{\sqrt{2}(1 - M)},\]

for all possible solutions \(x\) of (4).

Claim that there is \(\tau \in J\) such that \(|x(\tau)| < m + 1\), where \(m = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}\). Suppose that the claim is not true, so let \(x(t) \geq m + 1\) for all \(t \in J\). Then \(x(t) > \beta(t)\) for all \(t \in J\) and by the fact that \(\beta\) is an
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upper solution of (2), Eq. (4) becomes

\begin{align*}
x^{(n)}(t) &= (1 - \lambda) Mx(t) - \lambda F(t, x(t)) \\
&= Mx(t) - \lambda f(t, \beta(t)) - \lambda M\beta(t) \\
&\geq Mx(t) + \lambda \beta^{(n)}(t) - \lambda M\beta(t) \quad \text{a.e.}
\end{align*}

Thus

\begin{align*}
(x - \lambda \beta)^{(n)}(t) - M(x - \lambda \beta)(t) &\geq 0, \quad \text{a.e. } t \in J.
\end{align*}

Since \( x - \lambda \beta \in F^{n}_{2\pi} \) for all \( \lambda \in [0, 1] \), it follows from Lemma 1 that

\[ x(t) \leq \lambda \beta(t), \quad \text{for all } t \in J. \]

Thus

\[ |\beta(t)| < x(t) \leq \lambda \beta(t), \]

for all \( t \in J \) and \( \lambda \in [0, 1] \) and this is a contradiction. We may get a contradiction by a similar argument for the case \( x(t) \leq -m - 1, \) for all \( t \in J \), and the claim is verified. Now

\begin{align*}
|x(t)| &\leq |x(\tau)| + \int_{\tau}^{t} |x'(s)| \, ds \\
&\leq |x(\tau)| + 2\pi \|x'\|_{2} \\
&< m + 1 + \frac{\sqrt{6}\pi^{3}(M + \|h_{1}\|_{1})}{(1 - M)} \equiv M(h_{1}),
\end{align*}

and the \textit{a priori} estimate is complete. For degree computations, we reduce problem (4) to an equivalent operator form. Define \( L : D(L) \subset C^{0}_{2\pi}(J) \rightarrow L^{1}(J) \) by \( x \mapsto x^{(n)} \), where \( D(L) = W^{n,1}_{2\pi} \) and \( N_{\lambda} : C^{2}_{2\pi}(J) \rightarrow L^{1}(J) \) by

\[ N_{\lambda}x(\cdot) = -(1 - \lambda) Mx(\cdot) + \lambda F(\cdot, x(\cdot)) \]

so that (4) can be written as

\[ Lx + N_{\lambda}x = 0. \]
By the standard argument [5], we can easily check that \( L \) is a Fredholm operator of index 0 and \( N_\lambda \) is \( L \)-compact on \( \Omega \) for any bounded open subset \( \Omega \) in \( C^0_{2\pi}(J) \). Let \( \Omega_0 \) be an open bounded subset in \( C^0_{2\pi}(J) \) with

\[
\Omega_0 \supset \{ x \in C^0_{2\pi}(J) : \| x \|_\infty < M(h_1) \}.
\]

Then by the a priori estimate, \( Lx + N_\lambda x \neq 0 \) for \( x \in D(L) \cap \partial \Omega_0 \), and thus the coincidence degree \( D_L(L + N_\lambda, \Omega_0) \) is well-defined. Since the linear problem

\[
x^{(n)}(t) - Mx(t) = 0
\]

does not have any nontrivial \( 2\pi \)-periodic solutions, by homotopy invariance property and Proposition II.16 [5], we obtain

\[
\pm 1 = D_L(L - MI, \Omega_0) = D_L(L + N_0, \Omega_0) = D_L(L + N_1, \Omega_0).
\]

This implies that (3) has at least one solution in \( D(L) \cap \Omega_0 \) and the proof is complete.

3. MULTIPURITY RESULTS

In this section, we shall apply Theorem 1 of Section 2 to get multiplicity results of \( 2\pi \)-periodic solutions for higher order Ambrosetti–Prodi type problems. Let us consider Eq. (1.9):

\[
\begin{align*}
  &x^{(n)}(t) + f(t, x(t)) = s \quad \text{on } J, \\
  &x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1,
\end{align*}
\]

where \( s \) is a real parameter and \( f: J \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function. Throughout the remainder of this paper, sometimes without further comment, we shall assume the following condition; there exists \( M \in (0, A(n)) \) such that

\[
(f(t, x) - f(t, y))(x - y) \geq -M(x - y)^2, \quad \text{for a.e. } t \in J \text{ and } x, y \in \mathbb{R},
\]

\((H1')\)
where $A(n)$ is given in Lemma 1. We notice that (H1') implies the condition (H2) in Theorem 1 and if $f$ is continuous, then (H1') is equivalent to (H1).

**Theorem 2** Assume that there exist $s_1$ and $r(s_1) > 0$ such that

$$\text{ess sup}_{t \in J} f(t, 0) < s_1 < f(t, x)$$

for a.e. $t \in J$ and $x \in \mathbb{R}$ with $x \leq -r(s_1)$. Then there exits $s_0 < s_1$, possibly $s_0 = -\infty$ such that (1) has no solution for $s < s_0$ and at least one solution for $s \in (s_0, s_1]$.

**Proof** Let $s^* = \text{ess sup}_{t \in J} f(t, 0)$, then constant functions $\alpha \equiv -r(s_1)$ and $\beta \equiv 0$ are lower and upper solutions of (1), respectively. Thus by Theorem 1, Eq. (1) has a solution for $s = s^*$. We also see that if (1) has a solution $\bar{x}$ for $\bar{s} < s_1$, then (1) also has a solution for $s \in [\bar{s}, s_1]$, since $\bar{x}$ and $-r(s_1)$ are upper solution and lower solution of (1) for $s \in [\bar{s}, s_1]$, and $-r(s_1) \leq \bar{x}(t)$ for all $t \in J$ by necessary adjustment for $r(s_1)$. We complete the proof by taking $s_0 = \inf\{s \in \mathbb{R}: (1) \text{ has at least one solution}\}$.

For multiplicity results, we shall employ coincidence degree arguments. Define $L : D(L) \subset C^0_{2\pi}(J) \to L^1(J)$ by $x \mapsto x^{(n)}$, where $D(L) = W^{n, 1}_{2\pi}(J)$, and $N_s : C^0_{2\pi}(J) \to L^1(J)$ by

$$N_s x(\cdot) = f(\cdot, x(\cdot)) - s.$$

Then (1) can be equivalently written as

$$Lx + N_s x = 0,$$

and it is well-known that $L$ is a Fredholm operator of index 0 and $N_s$ is $L$-compact on $\bar{\Omega}$ for any bounded open subset $\Omega$ in $C^0_{2\pi}(J)$.

**Theorem 3** Assume that there exist $s_1$ and $r(s_1) > 0$ such that

$$\text{ess sup}_{t \in J} f(t, 0) < s_1 < f(t, x)$$

for a.e. $t \in J$ and $x \in \mathbb{R}$ with $|x| \geq r(s_1)$. Also assume that there exists $R = R(s_1, f) > 0$ such that every possible solution $x$ of (1),

\[\text{ess sup}_{t \in J} f(t, 0) < s_1 < f(t, x)\]
for \( s \leq s_1 \), satisfies

\[ \|x\|_{\infty} < R. \]  

Then there exists a real number \( s_0 < s_1 \) such that \((1, s)\) has

(i) no solution for \( s < s_0 \),
(ii) at least one solution for \( s = s_0 \),
(iii) at least two solutions for \( s \in (s_0, s_1] \).

Proof. We know that for \( s_0 \) given in Theorem 2, \((1, s)\) has no solution for \( s < s_0 \) and at least one solution for \( s \in (s_0, s_1] \).

First, we show that \( s_0 \) is finite. It follows from (3a) and \( f \) Carathéodory that

\[ f(t, x) \geq -|s_1| - h_r(t), \]

for a.e. \( t \) and all \( x \in R \), where \( h_r \) is the \( L^1 \)-function determined by \( r(s_1) \) in the definition of Carathéodory function. If \((1, s)\) has a solution \( x \), then

\[ s = \frac{1}{2\pi} \int_J f(t, x(t)) \, dt \geq -|s_1| - \frac{1}{2\pi} \|h_r\|_1. \]

Thus \( s_0 \geq -|s_1| - 1/2\pi \|h_r\|_1 > -\infty \).

Second, we show existence of the second solution of \((1, s)\) for \( s \in (s_0, s_1] \). Without loss of generality, let us assume that \( R > r(s_1) \). Let \( \Omega \) be an open bounded subset in \( C^0_{2\pi}(J) \) such that \( \Omega \supset \{ x \in C^0_{2\pi}(J) : \|x\|_{\infty} < R \} \); then by (3b), the coincidence degree \( D_L(L + N_s, \Omega) \) is well-defined. Since \((1, s)\) does not have solution for \( s > s_0 \), by the common argument of Ambrosetti-Prodi type problems [3,6], we get

\[ D_L(L + N_s, \Omega) = 0, \quad \text{for } s \leq s_1. \]  

Let \( s \in (s_0, s_1] \), \( \tilde{s} \in (s_0, s) \) and let \( \tilde{x} \) be a solution of \((1, \tilde{s})\) known to exist by Theorem 2. Then \( -R \) and \( \tilde{x} \) are lower and upper solutions of \((1, s)\) with \( -R < \tilde{x}(t) \), for all \( t \in J \). Let \( \Omega_1 = \{ x \in C^0_{2\pi}(J) : -R < x(t) < \tilde{x}(t), \ t \in J \} \), then \( \Omega_1 \subset \Omega \). By (3b) and Remark 1, solutions of \((1, s)\) never lie on \( \partial \Omega_1 \). Thus \( D_L(L + N_s, \Omega_1) \) is well-defined. To compute the degree, let us
consider a modified problem:

\[ x^{(n)}(t) + F(t, x(t)) = 0 \quad \text{a.e. on } J, \]
\[ x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n - 1, \]

where

\[ F(t, x) = \begin{cases} 
  f(t, \bar{x}(t)) - s - M(x - \bar{x}(t)), & \text{if } x > \bar{x}(t), \\
  f(t, x) - s, & \text{if } -R \leq x \leq \bar{x}(t), \\
  f(t, -R) - s - M(x + R), & \text{if } x < -R,
\end{cases} \]

and \( M \) is given in \((H1')\). By a similar argument as in the proof of Theorem 1, we get

\[ DL(L + NF, \Omega_0) = \pm 1, \]

for certain open bounded open subset \( \Omega_0 \) in \( C^0_{2\pi}(J) \), where \( NF \) is defined by \( NFx(\cdot) = F(\cdot, x(\cdot)) \), and we also know that all solution \( x \) of (5) must satisfy

\[ -R < x(t) < \bar{x}(t), \quad \text{for all } t \in J \]

so that \((1_s)\) is equivalent to (5) in \( \Omega_1 \). Therefore, by the excision and the additive properties of the coincidence degree together with \((3c)\), we get

\[ \pm 1 = DL(L + NF, \Omega_0) = DL(L + NF, \Omega_1) = DL(L + Ns, \Omega_1) \]

and

\[ DL(L + Ns, \Omega \setminus \bar{\Omega}_1) = \mp 1. \]

Consequently, \((1_s)\) has one solution in \( \Omega_1 \) and another in \( \Omega \setminus \bar{\Omega}_1 \). Since \( s \in (s_0, s_1] \) is arbitrary, the second part of the proof is complete.

Finally, the existence of at least one solution at \( s = s_1 \) can be proved through a limiting process based upon a priori boundedness of possible solutions as in [3].

**Theorem 4** Assume that \( f \) is a Carathéodory function and satisfies \((H)\) and \((H1')\). Moreover, assume that

\[ \text{ess} \sup_{t \in J} |f(t, 0)| < \infty. \]  \hspace{1cm} (3d)
Then there exists a real number $s_0$ such that (1,s) has

(i) no solution for $s < s_0$,
(ii) at least one solution for $s = s_0$,
(iii) at least two solutions for $s > s_0$.

Proof If $f$ satisfies (H), then (3a) in Theorem 3 is valid for arbitrary large $s_1$. Thus it suffices to show that all possible solutions of (1,s) for $s \leq s_1$ are uniformly bounded. Let $s \leq s_1$ and $x$ a solution of (1,s). Integrating both sides of (1,s) on $J$, we get

$$\int_J f(t, x(t)) \, dt = 2\pi s.$$

From the proof of Theorem 3, we know that

$$f(t, x) + |s_1| + h_r(t) \geq 0,$$

for a.e. $t$ and for all $x \in \mathbb{R}$. When $n$ is odd, Multiplying both sides of (1,s) by $x'$ and integrating on the period, we get for $p = (n + 1)/2$,

$$(-1)^p \|x^{(p)}\|_2^2 = \int_J f(t, x(t))x'(t) \, dt$$

$$= \int_J (f(t, x(t)) + |s_1| + h_r(t))x'(t) \, dt - \int_J h_r(t)x'(t) \, dt.$$

Thus

$$\|x^{(p)}\|_2^2 \leq \|x'\|_\infty \int_J (f(t, x(t)) + |s_1| + h_r(t)) \, dt + \|h_r\|_1 \|x'\|_\infty$$

$$\leq \|x'\|_\infty (2\pi(s + |s_1|) + 2\|h_r\|_1)$$

$$\leq \sqrt{\frac{2\pi}{3}} (2\pi |s_1| + \|h_r\|_1) \|x''\|_2, \quad \text{by Sobolev inequality}$$

$$\leq \sqrt{\frac{2\pi}{3}} M(s_1) \|x^{(p)}\|_2, \quad \text{by Wirtinger inequality},$$

where $M(s_1) = 2\pi |s_1| + \|h_r\|_1$. Thus

$$\|x^{(p)}\|_2 \leq \sqrt{\frac{2\pi}{3}} M(s_1),$$
and by Wirtinger inequality,
\[ \|x'\|_2 \leq \sqrt{\frac{2\pi}{3} M(s_1)}. \]

When \( n \) is even, multiplying both sides of \((1)\) by \( \tilde{x} \), integrating on \( J \) and doing similar calculations, we get for \( p = n/2 \),
\[
\|x^{(p)}\|_2^2 \leq \|\tilde{x}\|_\infty \int_J (f(t, x(t)) + |s_1| + h_r(t)) \, dt + \|h_1\|_1 \|\tilde{x}\|_\infty \\
\leq \|\tilde{x}\|_\infty (2\pi (|s_1|) + 2\|h_r\|_1) \\
\leq \sqrt{\frac{2\pi}{3}} (|s_1| + \|h_r\|_1) \|x'\|_2, \text{ by Sobolev inequality} \\
\leq \sqrt{\frac{2\pi}{3}} M(s_1) \|x^{(p)}\|_2, \text{ by Wirtinger inequality.}
\]

Thus we also get
\[ \|x'\|_2 \leq \sqrt{\frac{2\pi}{3} M(s_1)}. \]

We claim that for each possible solution \( x \) of \((1)\) and \( s \in (s_0, s_1) \), there is \( t_0 \in J \) such that \( |x(t_0)| < r(s_1) \). Suppose the claim is not true, then there exists a solution \( x \) such that
\[ x(t) \geq r(s_1), \text{ for all } t \in J. \]

So if \( x(t) \geq r(s_1) \) for all \( t \in J \), then by \((3a)\),
\[ f(t, x(t)) > s_1, \text{ a.e. in } t. \]

Thus
\[ s = \frac{1}{2\pi} \int_J f(t, x(t)) \, dt > s_1, \]
and this is a contradiction. Similarly, we may show a contradiction for the case \( x(t) \leq -r(s_1) \) and the claim is verified. Consequently,
\[
| x(t) | \leq | x(t_0) | + \int_{t_0}^t | x'(\tau) | \, d\tau < r(s_1) + 2\pi \| x' \|_2 \\
\leq r(s_1) + \frac{2\pi\sqrt{2\pi}}{\sqrt{3}} M(s_1). 
\]

The proof is complete.
Remark 2 If the function $h_r$ determined by $r$ in the definition of Carathéodory is of $L^\infty(J)$, then the condition (3d) in Theorem 4 is not necessary.

Corollary 1 If $f$ is continuous on $J \times \mathbb{R}$ and satisfies (H) and (H1), then there exists a real number $s_0$ such that (1) has

(i) no solution for $s < s_0$,

(ii) at least one solution for $s = s_0$.

(iii) at least two solutions for $s > s_0$.

Consider the equation

$$x^{(n)}(t) + g(x(t)) + h(t) = s \quad \text{on } J,$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-1,$$

Corollary 2 If $g : \mathbb{R} \to \mathbb{R}$ is continuous such that

$$\lim_{|x| \to \infty} g(x) = \infty,$$

and $g$ is also such that there exists $M \in (0, A(n))$ for which

$$(g(x) - g(y))(x - y) \geq -M(x - y)^2,$$

for all $x, y \in \mathbb{R}$. Then for any given $h \in L^\infty(J)$, there exists a real number $s_0$ such that (6) has

(i) no solution for $s < s_0$,

(ii) at least one solution for $s = s_0$,

(iii) at least two solutions for $s > s_0$.

References


