An Inequality on Solutions of Heat Equation*

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Let \( v(x, t) \) be the solution of the initial value problem for the \( n \) dimensional heat equation. Then, for any \( a \) and for any \( t_0 > 0 \), an inequality about \( v(a, t) \) and \( v(x, t_0) \) is obtained.

**Keywords:** Heat equation; Integral transform; Positive matrix; Reproducing kernel

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1. **INTRODUCTION**

For a positive integer \( n \), we consider the \( n \) dimensional heat equation

\[
\begin{align*}
\Delta v(x, t) & = \partial_t v(x, t), \quad x \in \mathbb{R}^n \text{ and } t > 0; \\
v(x, 0) & = F(x), \quad x \in \mathbb{R}^n
\end{align*}
\]

where \( \Delta \) is the \( n \) dimensional Laplacian and \( F \) is a member in the space \( L^2(\mathbb{R}^n) \) for the Lebesgue measure on \( \mathbb{R}^n \). Then, the solution is represented by

\[
v(x, t) = \left( \frac{1}{2\sqrt{\pi t}} \right)^n \int_{\mathbb{R}^n} F(\xi) \exp \left\{ -\frac{|x - \xi|^2}{4t} \right\} \, d\xi, \quad (1.2)
\]
Furthermore, from the expression (1.2) we know that the solution $v(x, t)$
can be holomorphically extended on the $n$ dimensional complex space
$\mathbb{C}^n$ with respect to the space variable $x$. For the time variable $t$ also,
$v(x, t)$ can be holomorphically extended on the right half plane
$D = \{ z | \Re z > 0 \}$ of the complex plane $\mathbb{C}$. These facts are found in [4,6].

In this paper, for any $a \in \mathbb{R}^n$ and for a fixed time $t_0$, we derive an
inequality which expresses the relation of $v(a, t)$ and $v(x, t_0)$. Our
inequality is the generalization of an inequality in [6] for the $n$
dimension.

**Theorem** For an initial values $F$ in $L^2(\mathbb{R}^n)$ let $v(x, t)$ be the solution of
the $n$ dimensional heat equation (1.1). Then, for any $a \in \mathbb{R}^n$ and for any
to > 0, the following inequality is valid:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_t v(a, z)|^2 x^{n/2} \, dx \, dy \leq C(n, t_0) \int_{\mathbb{C}^n} |v(w, t_0)|^2 \exp \left( -\frac{|\tau|^2}{2t_0} \right) \, d\lambda \, d\tau,$$

where $z = x + iy$ ($x, y \in \mathbb{R}$), $w = \lambda + i\tau$ ($\lambda, \tau \in \mathbb{R}^n$). Moreover, the equality
holds if and only if $F$ is a member in $M(n, a)$. Here, $C(n, t_0) =
n\Gamma(n/2)/(2^{n+1}\pi^{n-1}t_0^{n/2})$ and $M(n, a)$ is the closure of the space spanned
linearly by

$$\left\{ f(\xi) = e^{-|\xi-a|^2} \text{ on } \mathbb{R}^n \mid a \in D \right\}$$

in $L^2(\mathbb{R})$.

**2. SOME HOLOMORPHIC FUNCTION SPACES**

We let $K(z, u)$ be the Bergman kernel on the domain $D$ with respect to
the measure $dx \, dy/\pi$. It is explicitly represented by $K(z, u) = 1/(z + \bar{u})^2$.
For any $q \geq 1$, we consider the Hilbert space

$$H_q = \left\{ f: \text{holomorphic in } D \mid \right.$$}

$$\left. \|f\|^2_{H_q} = \frac{1}{\pi \Gamma(2q - 1)} \int_{\mathbb{C}^n} |f(z)|^2 K(z, z)^{1-q} \, dx \, dy < \infty, \right.$$}

$$z = x + iy \}. \right.$$
Then, the kernel function
\[ K_q(z, u) = \Gamma(2q)K(z, u)^q, \quad (z, u) \in D \times D, \]
is the reproducing kernel of \( H_q \) in the following sense: for any \( z \in D \), \( K(\cdot, z) \) is the member in \( H_q \) and every member \( f \) in \( H_q \) is represented by
\[ f(z) = \langle f, K_q(\cdot, z) \rangle_{H_q}, \quad z \in D, \]
where \( \langle \cdot, \cdot \rangle_{H_q} \) is the inner product in the Hilbert space \( H_q \) (refer to [2,3]). Meanwhile, the kernel function \( K_q \) can be represented by
\[ K_q(z, u) = \int_0^\infty e^{-\xi \zeta} e^{-\xi_q \xi} e^{2q-1} d\xi, \quad z, u \in D, \quad (2.1) \]
and the right hand side of (2.1) converges for all \( q > 0 \). Hence, for any \( q \) with \( 0 < q < 1 \), the function \( K_q \) also determines the \( H_q \) that admits the reproducing kernel \( K_q(z, u) \) (see [1,7]). For any \( q > 0 \), we denote
\[ K_q(z, u) = \Gamma(2q)K(z, u)^q, \quad z, u \in D, \]
and we also consider the Hilbert space
\[ A_q = \left\{ g: \text{holomorphic in } D \mid \right. \]
\[ \|g\|^2_{A_q} = \frac{1}{\pi \Gamma(2q + 1)} \int_D \int_D |g'(z)|^2 K(z, z)^{-q} dx dy < \infty, \]
\[ \lim_{x \to \infty} g(x) = 0 \} . \]
Since the mapping \( f \mapsto f' \) is the isometry from \( H_q \) onto \( H_{q+1} \), \( H_q = A_q \), and \( K_q(z, u) \) is the reproducing kernel of \( A_q \) (see [3]).

3. PROOF OF THEOREM

Following the theory of generalized integral transforms [5], we prove our theorem. First, for \( a = 0 \), we consider the integral transform
\[ \mathcal{H}F(z) = \left( \frac{1}{2\sqrt{\pi z}} \right)^n \int_{\mathbb{R}^n} F(\xi) \exp \left( -\frac{\xi^2}{4z} \right) d\xi = v(0, z), \quad z \in D, \]
For any $t_0 > 0$, we calculate the kernel form

$$T_n(z, u) = \left( \frac{1}{4\pi \sqrt{zu}} \right)^n \int_{\mathbb{R}^n} \exp \left( -\frac{\xi^2}{4z} - \frac{\xi^2}{4u} \right) d\xi$$

$$= \left( \frac{1}{2\sqrt{\pi}} \right)^n K(z, u)^{n/4}.$$

Since the function $T_n(z, u)$ is positive matrix on $D$, it determines the reproducing kernel Hilbert space $S_n$ (see [1, 7]). On the other hand, the space $S_n$ is characterized by

$$S_n = \left\{ f: \text{holomorphic in } D \mid \frac{2^{3n/2+1}\pi^{n/2-1}}{n\Gamma(n/2)} \int_D |f'(z)|^2 x^{n/2} dx dy < \infty \right\}.$$

Hence we have the norm inequality

$$\|v(0, z)\|_{S_n}^2 \leq \int_{\mathbb{R}^n} |F(\xi)|^2 d\xi. \quad (3.1)$$

For the orthogonal complement $N^\perp$ of the null space

$$N = \bigcap_{z \in D} \{ F \in L^2(\mathbb{R}^n) \mid \mathcal{H}F(z) = 0 \},$$

the equality in (3.1) holds if and only if $F$ is a member in $N^\perp$. In fact, $N^\perp$ is the closure of the space in $L^2(\mathbb{R}^n)$ which is linearly spanned by members of the family

$$\{ G(\xi) = \exp(-\alpha|\xi|^2) \mid \alpha \in D \},$$

and so $N^\perp = M(n, 0)$. From [4], the norm equality

$$\left( \frac{1}{2\pi t_0} \right)^{n/2} \int_{\mathbb{R}^n} |v(w, t_0)|^2 \exp \left( -\frac{r^2}{2t_0} \right) d\lambda dr = \int_{\mathbb{R}^n} |F(\xi)|^2 d\xi. \quad (3.2)$$

holds, and from (3.1) and (3.2) our inequality is obtained for $a = 0$. 
For any $a \in \mathbb{R}^n$, since
\[
 v(a, t) = \left( \frac{1}{2\sqrt{\pi t}} \right)^n \int_{\mathbb{R}^n} F(\xi) \exp \left( -\frac{|a - \xi|^2}{4t} \right) d\xi 
\]
\[
 = \left( \frac{1}{2\sqrt{\pi t}} \right)^n \int_{\mathbb{R}^n} F(\xi + a) \exp \left( -\frac{|\xi|^2}{4t} \right) d\xi,
\]
we have
\[
 \|v(a, t)\|_{S_n}^2 \leq \int_{\mathbb{R}^n} |F(\xi + a)|^2 d\xi = \int_{\mathbb{R}^n} |F(\xi)|^2 d\xi. \tag{3.3}
\]

From (3.2) and (3.3), the inequality (1.3) is valid. Meanwhile, the equality in (3.3) holds if and only if $F(\xi + a) \in M(n, 0)$. Therefore the proof has been completed.

References