MIXED PROBLEM WITH BOUNDARY INTEGRAL CONDITIONS FOR A CERTAIN PARABOLIC EQUATION

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ABSTRACT

The present article is devoted to a proof of the existence and uniqueness of a solution of a mixed problem with boundary integral conditions for a certain parabolic equation. The proof is based on an energy inequality and on the fact that the range of the operator generated by the problem is dense.

Key words: Parabolic Equation, Boundary Integral Conditions, Energy Inequality.

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1. Introduction

In the rectangle $Q = (0, b) \times (0, T)$, we consider the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} + (-1)^m a(t) \frac{\partial^{2m} u}{\partial x^{2m}} = f(x, t),$$

(1.1)

where $a(t)$ is bounded, $0 < a_0 \leq a(t) \leq a_1$, and $a(t)$ has the bounded derivative such that $0 < c_0 \leq a'(t) \leq c_1$ for $t \in [0, T]$.

We adhere to equation (1.1) the initial condition

$$\ell u = u(x, 0) = \varphi(x)$$

(1.2)

and the boundary conditions

$$\int_0^b x^k \cdot u(x,t) dx = 0, \quad k = 0, 2m - 1.$$

(1.3)

The importance of problems with integral conditions has been pointed out by Samarskii [9]. Problems which combine local and integral condition for second order parabolic equations are investigated by the potential method [2, 7], by Fourier’s method [4-6], and by the energy inequalities method [1, 8, 10].

In this paper, the existence and uniqueness of a solution of problem (1.1)-(1.3) is proved. The proof is based on the method of energy inequalities, presented in [1]. Such problems have not been studied previously.
2. Preliminaries

First, we introduce the appropriate function spaces which will be used in the paper. We denote $B^m_2(0,b)$ by:

\[ B^m_2(0,b) = \begin{cases} L^2(0,b) & \text{for } m = 0, \\ \{u/\mathcal{T}^mu \in L^2(0,b)\} & \text{for } m \geq 1, \end{cases} \]

where $\mathcal{T}^mu = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)d\xi$, $m \geq 1$. For $m \geq 1$, the scalar product in $B^m_2(0,b)$ is defined by:

\[ (u,v)_{B^m_2(0,b)} = \int_0^b \mathcal{T}^mu \mathcal{T}^mv dx. \]

The associated norm is:

\[ \| u \|_{B^m_2(0,b)} = \| \mathcal{T}^mu \|_{L^2(0,b)} \text{ for } m \geq 1. \]

**Lemma 1:** For $m \in \mathbb{N}$, we have

\[ \| u \|_{B^m_2(0,b)}^2 \leq b^2 \| u \|_{B^{m-1}_2(0,b)}^2. \]

**Proof:** The Cauchy-Schwarz inequality gives

\[ |\mathcal{T}^mu|^2 \leq \left( \int_0^x |\mathcal{T}^{m-1} u(\xi,t)|^2 d\xi \right)^2 \leq \left( \int_0^x d\xi \right) \cdot \left( \int_0^x \left| \mathcal{T}^{m-1} u(\xi,t) \right|^2 d\xi \right) \]

\[ \leq x \cdot \int_0^x \left| \mathcal{T}^{m-1} u(\xi,t) \right|^2 d\xi \leq x \cdot \int_0^b \left| \mathcal{T}^{m-1} u(\xi,t) \right|^2 d\xi. \]

Therefore, we have

\[ \| u \|_{B^m_2(0,b)} \leq \int_0^b \left| \mathcal{T}^{m-1} u(\xi,t) \right|^2 d\xi \cdot \int_0^b x dx = b^2 \| u \|_{B^{m-1}_2(0,b)}. \]

**Corollary:** For $m \in \mathbb{N}$, we have

\[ \| u \|_{B^m_2(0,b)}^2 \leq \left( \frac{b^2}{2} \right)^m \| u \|_{L^2(0,b)}^2. \]

**Remark:** Inequalities (2.2) and (2.3) remain valid, if we replace the interval $(0,b)$ by a bounded region $\Omega$ of $\mathbb{R}^n$. It suffices to replace $b$ by $\text{meas}(\Omega)$ (measure of $\Omega$) in (2.2) and (2.3).

The space $B^{m,k}_2(Q)$ is the space with the finite norm

\[ \| u \|_{B^{m,k}_2(Q)}^2 = \int_0^T \| u(\cdot,t) \|_{B^{m}_2(Q)}^2 dt + \int_0^b \| u(\cdot,:) \|_{B^{k}_2(0,T)}^2 dx. \]
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The space $B^0_2(Q)$ coincides with $L^2(Q)$.

We associate with problem (1.1)-(1.3), the operator $L = (L, \ell)$ with domain denoted by $D(L) = E$. The operator $L$ is from $E$ to $F$; $E$ is Banach space of the functions $u \in L^2(0, b)$ satisfying (1.3), with the finite norm

$$
\| u \|_E^2 = \| \frac{\partial u}{\partial t} \|_{B^m_2(Q)}^2 + \| \frac{\partial^{2m} u}{\partial x^{2m}} \|_{B^m_2(Q)}^2 + \sup_{0 \leq \tau \leq T} \| u(x, \tau) \|_{L^2(0, b)}^2,
$$

(2.4)

where $F$ is the Hilbert space obtained by completing the space $B^m_2(Q) \times L^2(0, b)$ equipped with the norm

$$
\| \mathcal{F} \|_F^2 = \| f \|_{B^m_2(Q)}^2 + \| \varphi \|_{L^2(0, b)}^2, \mathcal{F} = (f, \varphi).
$$

(2.5)

Here, we assumed that the function $\varphi$ satisfies the conditions in the form (1.3), i.e.,

$$
\int_0^b x^k \cdot \varphi \, dx = 0, \quad k = 0, 2m - 1.
$$

(2.7)

3. Two-Sided A Priori Estimates

Theorem 1: The following a priori estimate

$$
\| Lu \|_F \leq c \| u \|_E
$$

(3.1)

holds for any function $u \in E$, where constant $c$ is independent of $u$.

Proof: Equation (1.1) implies that

$$
\| \ell u \|_{B^m_2(Q)} \leq 2 \left( \| \frac{\partial u}{\partial t} \|_{B^m_2(Q)}^2 + a_1^2 \| \frac{\partial^{2m} u}{\partial x^{2m}} \|_{B^m_2(Q)}^2 \right)
$$

(3.2)

and initial condition (1.2) yields

$$
\| Lu \|_{L^2(0, b)} \leq \sup_{0 \leq \tau \leq T} \| u(x, \tau) \|_{L^2(0, b)}.
$$

(3.3)

Combining inequality (3.2) with (3.3), we obtain (3.1) for $u \in E$, with $c = \max(2^{1/2}, 2^{1/2}a_1)$. □

Theorem 2: For any function $u \in E$, we have the inequality

$$
\| u \|_E \leq c \| Lu \|_F,
$$

(3.4)

where constant $c > 0$ does not depend on $u$.

Proof: We consider the scalar product in $L^2(Q^\tau)$, where $Q^\tau = (0, b) \times (0, \tau)$ and $0 \leq \tau \leq T$. Observe that

$$
2 \int_{Q^\tau} \left| \frac{\partial^m \partial u}{\partial t} \right|^2 \, dx \, dt + \int_0^b a(\tau) \left| u(x, \tau) \right|^2 \, dx
$$

$$
= 2R \left( Lu, (-1)^m \frac{\partial^m \partial u}{\partial t} \right)_{0, Q^\tau} + \int_0^b a(0) \left| \varphi \right|^2 \, dx + \int_{Q^\tau} a'(t) \left| u \right|^2 \, dx \, dt
$$

(3.5)
We estimate the first term on the right-hand side of (3.5). By applying an elementary inequality we have

$$2 \text{Re} \left( \mathcal{L}u, (-1)^m \tau^{2m} \frac{\partial u}{\partial t} \right)_{0, Q^r} \leq \| \mathcal{L}u \|_{B_{2}^{m,0}(Q^r)}^2 + \| \frac{\partial u}{\partial t} \|_{B_{2}^{m,0}(Q^r)}^2.$$  \hspace{1cm} (3.6)

From equation (1.1), we obtain

$$\frac{1}{4a_0^2} \| \frac{\partial^{2m} u}{\partial x^{2m}} \|_{B_{2}^{m,0}(Q^r)}^2 \leq \frac{1}{2} \| \frac{\partial u}{\partial t} \|_{B_{2}^{m,0}(Q^r)}^2 + \frac{1}{2} \| \mathcal{L}u \|_{B_{2}^{m,0}(Q^r)}^2.$$  \hspace{1cm} (3.7)

Therefore, by formulas (3.5)-(3.7),

$$\frac{1}{2} \| \frac{\partial u}{\partial t} \|_{B_{2}^{m,0}(Q^r)}^2 + \frac{1}{4a_0^2} \| \frac{\partial^{2m} u}{\partial x^{2m}} \|_{B_{2}^{m,0}(Q^r)}^2 + a_0 \| u(x, \tau) \|_{L^2(0,b)}^2 \leq \frac{3}{2} \| \mathcal{L}u \|_{B_{2}^{m,0}(Q^r)}^2 + a_1 \| \ell u \|_{L^2(0,b)}^2 + c_1 \| u \|_{L^2(Q^r)}^2.$$  \hspace{1cm} (3.8)

Applying Lemma 7.1 from [3] to the above inequality we get

$$\| \frac{\partial u}{\partial t} \|_{B_{2}^{m,0}(Q^r)}^2 + \| \frac{\partial^{2m} u}{\partial x^{2m}} \|_{B_{2}^{m,0}(Q^r)}^2 + \| u(x, \tau) \|_{L^2(0,b)}^2 \leq c_2 \left( \| \mathcal{L}u \|_{B_{2}^{m,0}(Q^r)}^2 + \| \ell u \|_{L^2(0,b)}^2 \right),$$

where

$$c_2 = \frac{\max(3/2, a_1)}{\min(1/2, 1/4a_0^2, a_0)} \exp(c_1 T).$$

Since the right-hand side of the above inequality does not depend on $\tau$, we can take the least upper bound of the left side with respect to $\tau$ from 0 to $T$. Thus, inequality (3.4) holds, where $c_2 = c^{1/2}$.

4. Solvability of the Problem

From inequality (3.1), it follows that operator $L: E \rightarrow F$ is continuous, while from inequality (3.4) it follows that the range of operator $L$ is closed in $F$ and, therefore, there is the continuous inverse operator $L^{-1}$ yielding the solution. In other words, this means that operator $L$ is a linear homeomorphism from the space $E$ on the closed set $R(L) \subset F$. To prove that problem (1.1)-(1.3) has a unique solution, it remains to show that $R(L) = F$.

**Theorem 3**: Let the conditions of Theorem 2 hold, and let the coefficient $a(t)$ have bounded derivatives up to the second order. Then, for any functions $f \in B_{2}^{m,0}(Q)$ and $\varphi \in L^2(0,b)$, there is a unique solution $u = L^{-1}f$ of problem (1.1)-(1.3), where $\mathcal{F} = (f, \varphi)$, and

$$\| u \|_E \leq c \left( \| f \|_{B_{2}^{m,0}(Q)} + \| \varphi \|_{L^2(0,b)} \right),$$

where constant $c$ is independent of $u$.

**Proof**: To prove Theorem 3, we need the following proposition.

**Proposition**: Let $D_0(L) = \{ u/u \in D(L), \ell u = 0 \}$ and let the conditions of Theorem 3 hold. If for $v \in B_{2}^{m,0}(Q)$ and for all $u \in D_0(L)$,
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\[(Lu, v)_{B^m_2,0(Q)} = 0, \quad (4.1)\]

then \(v\) vanishes almost everywhere on \(Q\).

**Proof of the Proposition:** Assume that relation (4.1) holds for any function \(u \in D_0(L)\). Using this fact we can express (4.1) in a special form. First define \(h\) by the formula

\[h = \int_{t}^{T} \frac{\partial}{\partial \tau} \left( a(\tau) \frac{\partial u}{\partial \tau} \right) d\tau.\]

Let \(\frac{\partial u}{\partial t}\) be a solution of

\[a(t) \frac{\partial u}{\partial t} = h \quad (4.2)\]

and let

\[D_s(L) = \{u/u \in D(L): u = 0 \text{ for } t \leq s\}. \quad (4.3)\]

We, now, have

\[v = -\frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right). \quad (4.4)\]

Relations (4.2) and (4.3) imply that \(u\) is in \(D_0(L)\). It possesses, in fact, a higher order of smoothness, and we have the following result:

**Lemma 2:** If the conditions of the proposition are met, then the function \(u\) defined by (4.2) and (4.3) has derivatives with respect to \(t\) up to the second order belonging to the space \(B^m_{2,0}(Q_s)\), where \(Q_s = (0, b) \times (s, T)\).

**Proof of Lemma 2:** To prove Lemma 2, we will use the following \(t\)-averaging operators: Let \(\omega \in C^\infty(\mathbb{R})\), \(\omega \geq 0\); \(\omega = 0\) in a neighborhood of \(t = 0\) and \(t = T\), and outside the interval \((0, T)\), and let \(\int \omega(t)dt = 1\). We consider the operators \(\rho_\varepsilon\) defined by the formula

\[(\rho_\varepsilon w)(x, t) = \frac{1}{\varepsilon} \int_{0}^{T} \omega \left( \frac{s-t}{\varepsilon} \right) w(x, s) ds \text{ for } w \in B^m_{2,0}(Q).\]

The above operators have the following properties:

**P1:** The function \(\rho_\varepsilon w \in C^\infty(Q)\) and it vanishes in a neighborhood of \(t = T\) if \(w \in B^m_{2,0}(Q)\), and \(\rho_\varepsilon u \in D_s(L)\) if \(u \in D_s(L)\).

**P2:** If \(w \in B^m_{2,0}(Q)\), then \(\| \rho_\varepsilon w - w \|_{B^m_{2,0}(Q)} \to 0\) when \(\varepsilon \to 0\), and \(\| \rho_\varepsilon w \|_{B^m_{2,0}(Q)} \leq \| w \|_{B^m_{2,0}(Q)}\).

**P3:** \(\frac{d}{dt} \rho_\varepsilon u = \rho_\varepsilon \frac{d}{dt} u\) for \(k = 1, 2\) if \(u \in D_s(L)\).

**P4:** If \(w \in B^m_{2,0}(Q)\) then,

\[\| \frac{\partial}{\partial t} (a(t) \rho_\varepsilon w - \rho_\varepsilon a(t)w) \|_{B^m_{2,0}(Q)} \to 0\] when \(\varepsilon \to 0\).

Proofs of properties P1-P4 are similar to the proofs of the corresponding properties obtained in [3] (see Lemma 9.1).\(
\)

Applying the operators \(\rho_\varepsilon\) and \(\frac{\partial}{\partial t}\) to equation (4.2), we obtain

\[a(t) \frac{\partial}{\partial t} (\rho_\varepsilon \frac{\partial u}{\partial t}) = \frac{\partial}{\partial t} \left( a(t) \rho_\varepsilon \frac{\partial u}{\partial t} - \rho_\varepsilon a(t) \frac{\partial u}{\partial t} \right) - a'(t) \rho_\varepsilon \frac{\partial u}{\partial t} + \frac{\partial}{\partial t} \rho_\varepsilon h.\]
It follows that

\[
\| a(t) \frac{\partial}{\partial t} \rho e \frac{\partial u}{\partial t} \|_{B^m_2,0(Q)} \leq c_3 \left( \| \rho e \frac{\partial u}{\partial t} \|_{B^m_2,0(Q)}^2 \right)
\]

\[
+ \| \frac{\partial}{\partial t} \rho e h \|_{B^m_2,0(Q)}^2 + \| \frac{\partial}{\partial t} \left( a(t) \rho e \frac{\partial u}{\partial t} - \rho e a(t) \frac{\partial u}{\partial t} \right) \|_{B^m_2,0(Q)}^2,
\]

where \( c_3 = \max(3c_1, 3) \).

By virtue of properties P1-P4 of the t-averaging operators and by inequality (2.3), we have

\[
\left( \| \frac{\partial^2 u}{\partial t^2} \|_{B^m_2,0(Q)} \leq c_4 \| \frac{\partial u}{\partial t} \|_{L^2(Q)}^2 + \| \frac{\partial}{\partial t} \rho e h \|_{B^m_2,0(Q)}^2 \right),
\]

where \( c_4 := \max \left( c_3 \frac{b^2}{a_0^2}, 1/a_0^2 \right) \). This yields the proof of Lemma 2. \(\square\)

Now, we will prove the proposition. Replace \( v \) in (4.1) by its representation (4.4). We have

\[
-2 \Re \left( \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)}
\]

\[
-2 \Re \left( (-1)^m a(t) \frac{\partial u}{\partial x} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)} = 0. \tag{4.5}
\]

We write the remaining two terms of (4.5) in the form:

\[
-2 \Re \left( \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)}
\]

\[
= \| a^{1/2}(s) \tau^m \frac{\partial u(x,s)}{\partial t} \|_{L^2(0,b)}^2 - \| a^{1/2}(t) \tau^m \frac{\partial u}{\partial t} \|_{L^2(Q)}^2,
\]

\[
-2 \Re \left( (-1)^m a(t) \frac{\partial^2 u}{\partial x^m} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)}
\]

\[
= 2 \| a(t) \frac{\partial u}{\partial t} \|_{L^2(Q)}^2 + \Re \left( a'(t) u(x,T), a(T) \bar{u}(x,T) \right)_{L^2(0,b)}
\]

\[
- \| a'(t) u \|_{L^2(Q)}^2 - \Re \left( a''(t) u, a(t) \bar{u} \right)_{L^2(Q)}.
\tag{4.7}
\]

Elementary calculations, starting from (4.6) and (4.7), yield the inequalities

\[
a_0 \| \frac{\partial u(x,s)}{\partial t} \|_{B^m_2(0,b)} \leq c_1 \| \frac{\partial u}{\partial t} \|_{B^m_2,0(Q)}^2 - 2 \Re \left( \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)},
\]

\[
2a_0^2 \| \frac{\partial u}{\partial t} \|_{L^2(Q)}^2 + a_0 c_0 \| u(x,T) \|_{L^2(0,b)}
\]

\[
\leq -2 \Re \left( (-1)^m a(t) \frac{\partial^2 u}{\partial x^m} \frac{\partial}{\partial t} \left( a(t) \frac{\partial u}{\partial t} \right) \right)_{B^m_2,0(Q)} + \left( 1/2a_1^2 + c_2^2 + 1/2c_3^2 \right) \| u \|_{L^2(Q)}^2
\]

where \( c_5 := \sup_{0 \leq t \leq T} | a''(t) | \).

Consequently,
Inequality (4.8) is the basic of our proof. To use (4.8), we note that constant $c_6$ is independent of $s$. However, function $u$ in (4.8) depends on $s$. To avoid this difficulty we introduce a new function $\theta$ by the formula

$$\theta(x, t) = \int_t^T \frac{\partial u}{\partial t} \, dt.$$ 

Then, $u(x, t) = \theta(x, s) - \theta(x, t)$, $u(x, T) = \theta(x, s)$, and we have

$$\| u \|_{L^2(Q_s)}^2 \leq 2 \left( \| \theta(x, t) \|_{L^2(Q_s)}^2 + (T - s) \| \theta(x, s) \|_{L^2(0, b)}^2 \right).$$

Hence, if $s_0 > 0$ satisfies $0 < 2c_6(T - s_0) \leq 1/2$, then (4.8) implies that

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q_s)}^2 + \left\| \frac{\partial u(x, s)}{\partial t} \right\|_{B^m(0, b)}^2 + \| \theta(x, s) \|_{L^2(0, b)}^2 \leq 4c_6 \left( \left\| \frac{\partial u}{\partial t} \right\|_{B^m(0, b)}^2 + \| \theta(x, t) \|_{L^2(Q_s)}^2 \right),$$

for all $s \in [T - s_0, T]$.

We denote the sum of the two terms on the right of (4.9) by $\beta(s)$. Hence, we obtain

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q_s)}^2 \cdot \frac{d\beta(s)}{ds} \leq 4c_6 \beta(s),$$

and, consequently,

$$- \frac{d}{ds} (\beta(s) \exp(4c_6s)) \leq 0. \quad (4.10)$$

Integrating (4.10) over $(s, T)$ and taking into account that $\beta(T) = 0$, we obtain

$$\beta(s) \exp(4c_6s) \leq 0. \quad (4.11)$$

It follows from (4.11), that $v = 0$ almost everywhere on $Q_{T - s_0}$. Proceeding this way step by step along the rectangle with side $s_0$, we prove that $v = 0$ almost everywhere on $Q$. This completes the proof of the proposition. \hfill \Box

Now, we will prove Theorem 3. For this purpose it is sufficient to prove that the range $R(L)$ of $L$ is dense in $F$.

Suppose that, for some $V = (v, v_0) \in \perp R(L)$,

$$(\ell u, v)_{B^m(0, b)} + (\ell u, v_0)_{L^2(0, b)} = 0. \quad (4.12)$$

We must prove that $V = 0$. Putting $u \in D_0(L)$ into (4.12) we obtain

$$(\ell u, v)_{B^m(0, b)} = 0, \quad u \in D(L).$$
Hence, the proposition implies that \( v = 0 \). Thus, (4.12) takes the form

\[
(\ell u, v_0)_{L^2(0,b)} = 0, \quad u \in D(L).
\]

Since the range of operator \( \ell \) is everywhere dense in \( L^2(0,b) \), the above relation implies that \( v_0 = 0 \). Hence, \( V = 0 \). This proves Theorem 3. \( \square \)

References