ON WEAK SOLUTIONS OF RANDOM
DIFFERENTIAL INCLUSIONS

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ABSTRACT

In the paper we study the existence of solutions of the random differential inclusion
\[ \dot{x}_t \in G(t, x_t) \quad P\cdot \text{a.e.} \quad x_0 \overset{d}{=} \mu, \]
where \( G \) is a given set-valued mapping value in the space \( K^n \) of all nonempty, compact and convex subsets of the space \( \mathbb{R}^n \), and \( \mu \) is some probability measure on the Borel \( \sigma \)-algebra in \( \mathbb{R}^n \). Under certain restrictions imposed on \( F \) and \( \mu \), we obtain weak solutions of problem (I), where the initial condition requires that the solution of (I) has a given distribution at time \( t = 0 \).

Key words: Set-Valued Mappings, Hukuchara’s Derivative, Aumann’s Integral, Tightness and Weak Convergence of Probability Measures.

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1. Preliminaries

Problems of existence of solutions of differential inclusions were studied by many. In particular, random cases were considered in [3], [5], [7]. This work deals with the inclusion with a purely stochastic initial condition. First, we recall several notions and results needed in the sequel. Let \( K_c(S) \) be the space of all nonempty compact and convex subsets of a metric space \( S \) equipped with the Hausdorff metric \( H \) (see e.g., [1], [4]):
\[ H(A, B) = \max(\bar{H}(A, B), \bar{H}(B, A)); A, B \in K_c(S), \]
where \( \bar{H}(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(a, b). \) By \( \| A \| \) we denote the distance \( H(A, 0) \). For \( S \) being a separable Banach space, \( (K_c(S), H) \) is a polish metric space.

Let \( I = [0, T], \ T > 0. \) For a given multifunction \( G:I \to K_c(S) \) by \( D_H G(t_0), \) we denote its Hukuchara derivative at the point \( t_0 \in I \) (see e.g., [2], [9]) by the limits (if they exist in \( K_c(S) \))
\[ \lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \lim_{h \to 0^+} \frac{F(t_0) - F(t - h)}{h}, \]
both equal to the same set \( D_H F(t_0) \in K_c(S). \)

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For $S = \mathbb{R}^n$ and $K^n = K_c(\mathbb{R}^n)$, we denote by $C_I = C(I, K^n)$ the space of all $H$-continuous set-valued mappings. In $C_I$ we consider a metric $\rho$ of uniform convergence

$$\rho(F, G) = \sup_{0 \leq t \leq T} H(X(t), Y(t)), \quad \text{for} \quad X, Y \in C_T.$$ 

Then $C_I$ is a polish metric space.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space. We recall now the notion of a multivalued stochastic process. The family of set-valued mappings $X = (X_t)_{t \geq 0}$ is said to be a multivalued stochastic process if for every $t \geq 0$, the mapping $X_t: \Omega \to K^n$ is measurable, i.e., $\tilde{X}_t(U) := \{\omega: X_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$, for every open set $U \subseteq \mathbb{R}^n$ (see e.g., [1, 4]). It can be noted that $U$ can be also chosen as closed or Borel subset. We restrict our interest to the case when $0 \leq t \leq T$, $T > 0$. If the mapping $t \to X_t(\omega)$ is continuous ($H$-continuous) with probability on $(P.1)$, then we say that the process $X$ has continuous “paths.”

Let us notice that the set-valued stochastic process $X$ can be though as a random element $X: \Omega \to C_I$. Indeed, it follows immediately from [3] and from the fact that the topology of the uniform convergence and the compact-open topology in $C_I$ are the same.

**Definition 1:** A probability measure $\mu$ (on $C_I$) is a distribution of the set-valued process $X = (X_t)_{0 \leq t \leq T}$ if one has $\mu(A) = P(\tilde{X}(A))$ for every Borel subset $A$ from $C_I$.

A distribution of $X$ will be denoted by $P^X$.

**Definition 2:** A set-valued mapping $F: I \times K^n \to K^n$ is said to be an integrably bounded of the Caratheodory type if:

1) there exists a measurable function $m: I \to \mathbb{R}_+$ such that $\int_0^T m(t)dt < \infty$ and $\|F(t, \cdot)\| \leq m(t)$ t-a.e., $A \in K^n$.

2) $F(t, \cdot)$ is $H$-continuous t.a.e.

3) $F(\cdot, A)$ is a measurable multifunction for every $A \in K^n$.

Let us consider now the multivalued random differential equation:

$$D_H X_t = F(t, X_t) \quad P.1, t \in [0, T] \text{-a.e.}$$ 

(II)

$$X_0 \overset{d}{=} \mu$$

where the initial condition requires that the set-valued solution process $X = (X_t)_{t \in I}$ has a given distribution $\mu$ at the time $t = 0$. By a weak solution of (II) we understand a system $(\Omega, \mathcal{F}, P(X_t)_{t \in I})$ where $(X_t)_{t \in I}$ is a set-valued process on some probability space $(\Omega, \mathcal{F}, P)$ such that (II) is met.

We state the following theorem (see e.g. [6]).

**Theorem 1:** Let $F: I \times K^n \to K^n$ be an integrably bounded set-valued function of the Caratheodory type and let $\mu$ be an arbitrary probability measure on the space $K^n$. Then there exists a weak solution of (II).

2. Weak Solutions of Random Differential Inclusions

As an application of Theorem 1, we show the existence of a weak solution of the random differential inclusion

$$\dot{x}_t = G(t, x_t) \quad P.1, t \in [0, T] \text{-a.e.}$$

$$x_0 \overset{d}{=} \mu.$$

(I)
The weak solution of (I) is understood similarly as above, where $\mu$ is now a given probability measure on $\mathbb{R}^n$.

Let $\mathcal{T}_0$ denote the family of nonempty open subsets of $\mathbb{R}^n$, and let $C = \{C_V: V \in \mathcal{T}_0\}$, where $C_V = \{K \in K^n: K \cap V \neq \emptyset\}$. Then we have that $\mathbb{B}^n = \sigma(C)$ (see e.g. Proposition 3.1 [4]), where $\mathbb{B}^n$ is a Borel $\sigma$-field induced by the metric space $(K^n, H)$.

**Lemma 1:** The following hold true:

i) $K^n \subseteq C$,

ii) if $A_1, A_2, \ldots \in C$ then $\bigcup_{n=1}^{\infty} A_n \subseteq C$,

iii) if $C_{V_1} \subseteq C_{V_2} \subseteq \ldots$ then $V_1 \subseteq V_2 \subseteq \ldots$.

**Proof:** The property i) is obvious. Let $V_1, V_2, \ldots \in \mathcal{T}_0$ be such that $A_n = C_{V_n}$ for $n = 1, 2, \ldots$. To establish ii), let us observe that $\bigcup_{n=1}^{\infty} A_n = C \bigcup_{n=1}^{\infty} V_n$.

Let us suppose that iii) does not hold. Then for some $k \geq 1$, $V_k \not\subseteq V_{k+1}$. Hence there exists a point $x \in V_k$ such that $x \not\in V_{k+1}$. But then $\{x\} \subseteq C_{V_k}$ and $\{x\} \not\subseteq C_{V_{k+1}}$ contradicts to $C_{V_k} \subseteq C_{V_{k+1}}$.

To obtain our main result we need the following lemma:

**Lemma 2:** If $\mu$ is a probability measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^n)$, then there exists a probability measure $\tilde{\mu}$ on the space $K^n$ such that $\tilde{\mu}(C_V) = \mu(V), V \in \mathcal{T}_0$.

**Proof:** Let $C$ be the family generating Borel $\sigma$-field $\mathbb{B}^n$. We define a set-function $\nu$ on $C$ by $\nu(C_V) = \mu(V)$. Let us observe that $\nu$ is well-defined. Indeed, if $C_{V_1} = C_{V_2}$ and $\mu(V_1) \neq \mu(V_2)$ then $V_1 \not\subseteq V_2$. Hence $V_1 \backslash V_2 \neq \emptyset$ or $V_2 \backslash V_1 \neq \emptyset$. Without loss of generality we may assume the first case. Then there exists $x \in V_2$ such that $x \not\in V_1$. But then $\{x\} \subseteq C_{V_2}$ and $\{x\} \not\subseteq C_{V_1}$ which contradicts with an equality $C_{V_1} = C_{V_2}$. Similarly, it can be shown that if the sets $C_{V_1}$ and $C_{V_2}$ are disjoint, then the sets $V_1, V_2$ have the same property too. Hence we get $\nu(C_{V_1} \cup C_{V_2}) = \nu(C_{V_1}) + \nu(C_{V_2})$ for disjoint $C_{V_1}$ and $C_{V_2}$. From Lemma 1 we conclude that, if $C_{V_1} \subseteq C_{V_2} \subseteq \ldots$, then

$$\bigcup_{n=1}^{\infty} C_{V_n} \subseteq C \text{ and } \nu \left( \bigcup_{n=1}^{\infty} C_{V_n} \right) = \lim_{n \to \infty} \nu(C_{V_n}).$$

Moreover, $\nu(K^n) = 1$. Finally let us observe that $\nu$ is $\sigma$-subadditive. Next we define another set function $\tilde{\nu}$ as follows:

$$\tilde{\nu}(A) = \inf \{\nu(D): A \subseteq D, D \in C\}, \ A \subseteq K^n.$$  

Standard calculations show that $\tilde{\nu}$ is an outer measure on $K^n$. Thus from the Caratheodory Theorem, $\tilde{\nu}$ is a probability measure on the $\sigma$-field of $\tilde{\nu}$-measurable subsets in $K^n$. Setting $\tilde{\mu} = \tilde{\nu} | \mathbb{B}^n$, we obtain a desired probability measure.

We now present the following existence theorem.

**Theorem 2:** Let us suppose that $G: I \times \mathbb{R}^n \to K^n$ is an integrably bounded multifunction of the Caratheodory type. Then for any probability measure $\mu$ on $\mathbb{R}^n$, there exists a weak solution of problem (I).

**Proof:** Lemma 2 yields the existence of a probability measure $\tilde{\mu}$ on the metric space $(K^n, H)$ with the property: $\tilde{\mu}(C_V) = \mu(V), V \in \mathcal{T}_0$. Let $F: I \times K^n \to K^n$ be a multifunction defined by $F(t, A) = \bar{c}G(t, A)$, for $A \in K^n$. Hence from Lemma 1.1 [9], the set-valued mapping $F$ is integrably bounded of the Caratheodory type too. Consequently, by Theorem 1, there exists a probability space $(\Omega, \mathcal{F}, P)$ and the set-valued stochastic process $X = (X_t)_{0 \leq t \leq T}$ (on it) with
continuous "paths" and with values in $K^n$ which is a weak solution of the equation

$$D_H X_t = F(t, X_t) \quad P.1, \ t \in [0, T]\text{-a.e.}$$

$$X_0 \overset{d}{=} \hat{\mu}.$$  

From Kuratowski and Ryll-Nardzewski Selection Theorem [4] we can choose $\xi: \Omega \rightarrow \mathbb{R}^n$ as a measurable selection of $X_0$. Then by Theorem 4 [5] (see also [3]), there exists a stochastic process $x = (x_t)_{0 \leq t \leq T}$ as a selection of $X$ that is a solution (in strong sense) of the random differential inclusion:

$$\dot{x}_t \in G(t, x_t) \quad P.1, \ t \in [0, T]\text{-a.e.}$$

$$x_0 \in U \quad P.1,$$

where $U(\omega) = \{\xi(\omega)\}$ for $\omega \in \Omega$.

To complete the proof, it is sufficient to show that $x_0 \overset{d}{=} \mu$. Let us notice that $\{\omega: x_0(\omega) \in V\} = \{\omega: \xi(\omega) \in V\} \subset \{\omega: X_0 \cap V \neq \emptyset\}, \ V \in \mathcal{F}_0$. Because of $X_0 \overset{d}{=} \hat{\mu}$ and $\hat{\mu}(C_V) = \mu(V)$ we have

$$P^x_0(V) \leq \mu(V).$$  \hspace{1cm} (*)&

Using regularity properties of probability measures (on a separable metric space) (see e.g., Th. 1.2 [8]), we have that

$$P^x_0(B) = \inf\{P^x_0(V): B \subset V, V \in \mathcal{F}_0\}$$

and $\mu(B) = \inf\{\mu(V): B \subset V, V \in \mathcal{F}_0\}$ for every Borel subset $B$ of $\mathbb{R}^n$. Hence from inequality (*) we get $P^x_0(B) \leq \mu(B)$. But $P^x_0$ and $\mu$ are probability measures. Therefore they have to be equal.

References


