IN Variant Probabilities for Feller-Markov Chains

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Abstract
We give necessary and sufficient conditions for the existence of invariant probability measures for Markov chains that satisfy the Feller property.

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1. Introduction

The existence of invariant probabilities for Markov chains is an important issue for studying the long-term behavior of the chains and also for analyzing Markov control processes under the long-run expected average cost criterion. Inspired by the latter control problems, we present in this paper, two necessary and sufficient conditions for the existence of invariant probabilities for Markov chains that satisfy the Feller property. Our study extends previous results using stronger assumptions, such as the strong Feller property in Beneš [1], nondegeneracy assumptions (see condition (2) in Beneš [2]), and a uniform countable-additivity hypothesis in Liu and Susko [8]. As can be seen in the references, it is also worth noting that there are many reported results providing (only) sufficient conditions for the existence of invariant measures; in contrast however, our conditions are also necessary.

The setting for this paper is specified in Section 2, and our main result is presented in Section 3.
2. Notation and Definitions

Let $X$ be a $\sigma$-compact metric space, and let $\{x_t, t = 0, 1, \ldots\}$ be an $X$-valued Markov chain with time-homogeneous kernel $P$, i.e.,

$$P(B \mid x) = \text{Prob}(x_{t+1} \in B \mid x_t = x) \forall t = 0, 1, \ldots, x \in X, B \in \mathcal{B}(X),$$

where $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra of $X$. A probability measure (p.m.) $\mu$ on $\mathcal{B}(X)$ is said to be invariant for $P$ if

$$\mu(B) = \int P(B \mid x)\mu(dx) \quad \forall B \in \mathcal{B}(X).$$

Here, we give necessary and sufficient conditions for the existence of invariant p.m.'s when $P$ satisfies the Feller property:

$$\int \mu(B) = \int \mathbb{P}(B \mid x)\mu(dx) \quad \forall B \in \mathcal{B}(X).$$

Theorem: If $P$ satisfies the Feller property, then the following conditions (a), (b), and (c) are equivalent:

3. The Theorem

If $\nu$ is a p.m. on $X$, $E_\nu(\cdot)$ stands for the expectation given the “initial distribution” $\nu$. Theorem: If $P$ satisfies the Feller property, then the following conditions (a), (b), and (c) are equivalent:
(a) There exists a p.m. \( \nu \) and a moment \( v \) such that
\[
\limsup_{n \to \infty} J_n(\nu) < \infty,
\]
where \( J_n(\nu) := n^{-1} E_{\nu} \left[ \sum_{i=0}^{n-1} v(x_i) \right] \).

(b) There exists a p.m. \( \nu \) and a moment \( v \) such that
\[
\limsup_{\alpha \to 1} V_\alpha(\nu) < \infty,
\]
where \( V_\alpha(\nu) := (1 - \alpha) E_{\nu} \left[ \sum_{t=0}^{\infty} \alpha^t v(x_t) \right] \).

(c) There exists an invariant probability for \( P \).

**Proof:** We will show that \( (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) \).

(a) implies (b): This follows from a well-known Abelian theorem (e.g., see Sznajder and Filar [11], Theorem 2.2), which states that
\[
\limsup_{\alpha \to 1} V_\alpha(\nu) \leq \limsup_{n \to \infty} J_n(\nu).
\]
(Since a direct proof that (a) implies (c) is surprisingly simple, it will also be included here; see Remark 1 below.)

(b) implies (c): Suppose that (b) holds and for each \( c \in (0, 1) \), let \( \mu_\alpha \) be the probability measure on \( X \) defined as
\[
\mu_\alpha(B) := (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t \int_X P^t(B \mid z) \nu(dz), \quad B \in \mathcal{B}(X).
\]
Then we may write \( V_\alpha(\nu) = \int \nu d\mu_\alpha \). Let \( \{\alpha_n\} \) be a sequence in \( (0, 1) \) such that \( \alpha_n \uparrow 1 \) and, by (b),
\[
\limsup_{\alpha \to 1} V_\alpha(\nu) = \lim_{n \to \infty} V_{\alpha_n}(\nu) = \lim_{n \to \infty} \int \nu d\mu_{\alpha_n} < \infty.
\]
By the lemma in Section 2, \( \{\mu_{\alpha_n}\} \) is tight and therefore, by Prohorov's Theorem, \( \{\mu_{\alpha_n}\} \) contains a weakly convergent subsequence, which we denote by \( \{\mu_{\alpha_n}\} \) again; that is, there exists a probability measure \( \mu \) on \( X \) such that
\[
\lim_{n} \int \nu d\mu_{\alpha_n} = \int \nu d\mu \quad \forall \nu \in C(X).
\]
We claim that \( \mu \) is invariant for \( P \).

To see this, first note that by the Markov property, we may write \( \mu_\alpha \) as
\[
\mu_\alpha(B) = (1 - \alpha) \nu(B) + \alpha \int P(B \mid x) \mu_\alpha(dx) \quad \forall \alpha \in (0, 1), \quad B \in \mathcal{B}(X).
\]
Hence, for any \( u \in C(X) \),
\[
\int u \nu d\mu = (1 - \alpha) \int u(x) \nu(dx) + \alpha \int \int u(y) P(dy \mid x) \mu_\alpha(dx),
\]
and furthermore, note that by the Feller property (1), \( \int u(y)P(dy \mid \cdot) \) is in \( C(X) \). Thus, replacing \( \alpha \) by \( \alpha_n \) and letting \( n \to \infty \), we obtain
\[
\int ud\mu = \int \int u(y)P(dy \mid x)\mu(dx).
\] (2)

Finally, since \( u \in C(X) \) was arbitrary, we conclude from (2) that \( \mu \) is invariant for \( P \).

\( (c) \ implies \ (a) \): Let \( \nu \) be an invariant probability for \( P \), and let \( \{K_n\} \) be an increasing sequence of compact sets such that \( K_n \cap X \) and \( \nu(K_{n+1} - K_n) < 1/n^3 \), \( n = 1, 2, \ldots \). (Here we have used the fact that every p.m. on a \( \sigma \)-compact metric space is tight; see [3], p. 9.) Define a function \( v(\cdot) := 0 \) on \( K_1 \) and \( v(x) := n \) for \( x \in K_{n+1} - K_n, n \geq 1 \). Then \( v \) is a moment and
\[
\limsup_{n \to \infty} J_n(\nu) = \int v(x)\nu(dx) \leq \sum_{n=1}^{\infty} n^{-2} < \infty.
\]

Remark 1: We will prove directly that (a) implies (c). Suppose that (a) holds and for every \( n = 1, 2, \ldots \), let \( \mu_n \) be the probability measure on \( X \) defined as
\[
\mu_n(B) := n^{-1} \sum_{t=0}^{n-1} \int P^t(B \mid z)\nu(dz), \quad B \in \mathfrak{B}(X),
\]
so that we may rewrite the condition in (a) as
\[
\limsup_{n \to \infty} \int v d\mu_n < \infty.
\]
Hence, by the lemma in Section 2, \( \{\mu_n\} \) has a subsequence \( \{\mu_{n_i}\} \) which converges weakly to some probability measure \( \mu \). We will show that (cf. (2))
\[
\int Lu(x)d\mu(dx) = 0 \quad \forall u \in C(X),
\] (3)
where \( Lu(x) := \int u(y)P(dy \mid x) - u(x) \), thus showing that \( \mu \) is invariant for \( P \).

Indeed, for any bounded measurable function \( u \) on \( X \), the sequence
\[
M_n(u) := u(x_n) - \sum_{t=0}^{n-1} Lu(x_t), \quad n = 1, 2, \ldots,
\]
with \( M_0(u) := u(x_0) \), is a martingale, which implies
\[
E_\nu[M_n(u)] = E_\nu[M_0(u)] \forall n,
\]
i.e.,
\[
E_\nu[u(x_n)] - n \int Lu(x)d\mu_n(dx) = \int u(x)\nu(dx).
\]
Finally, let \( u \) be in \( C(X) \); replace \( n \) by \( n_i \); multiply by \( 1/n_i \); and then let \( i \to \infty \) to get (3).

Remark 2: In [8], it is shown that
\[
\sup_{t \geq 1} \int g(y)P^t(dy \mid x)\nu(dx) < \infty
\]
for some moment \( g \) and initial p.m. \( \nu \), is also a necessary and sufficient condition for existence of invariant probabilities provided that the Markov chain satisfies the uniform countable-additivity property.
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\[ \lim_{A \to \emptyset} \sup_{x \in K} P(A \mid x) = 0 \]  

for every compact set \( K \) in \( x \).

Note that (4) is stronger than our condition \((a)\) and that (5) implies: For every compact set \( K \subset X \), the family of p.m.'s \( \{P(\cdot \mid x)\}_{x \in K} \) is tight.

**Remark 3:** It is worth noting that the theorem still holds if we replace “\( \limsup \)” by “\( \liminf \)” in both conditions \((a)\) and \((b)\). Now, \((b) \Rightarrow (a)\) by a well-known Abelian theorem [11]. With similar arguments as in Remark 1, \((a) \Rightarrow (c)\). We finally prove \((c) \Rightarrow (b)\) by exhibiting the same moment function \( v \) and show that

\[ \liminf_{\alpha \downarrow 1} V_\alpha(\nu) = \liminf_{\alpha \downarrow 1} (1 - \alpha)E\nu \sum_{t=0}^{\infty} \alpha^t u(x_t) = \int v(x) \nu(dx) \leq \sum_{t=0}^{\infty} n^{-2} < \infty. \]

In conclusion, we mention that the theorem can be extended in the obvious way to continuous-time Markov processes, as in [2]. Conditions for uniqueness and ergodicity of invariant measures can be found, for instance, in [4, 7, 10] and references therein.

References