RANDOM FIXED POINTS OF WEAKLY INWARD OPERATORS IN CONICAL SHELLS

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ABSTRACT

Conditions for random fixed points of condensing random operators are obtained and subsequently used to prove random fixed point theorems for weakly inward operators in conical shells.

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1. Introduction

The fixed point theorems play a very important role in the questions of existence, uniqueness and successive approximations for various type of equations. The random fixed point theorems are useful tools for solving various problems in the theory of random equations, which form a part of random functional analysis. In recent years, there has been exciting interaction between analysis and probability theory, furnishing a rich source of problems for analysis. Bharucha-Reid [2] proved the random version of Schauder's Fixed Pint Theorem. Random fixed point theory has further developed rapidly in recent years; see e.g. Itoh [4], Sehgal and Waters [8], Sehgal and Singh [7], Papageorgiou [6], Lin [5], Xu [10], Tan and Yuan [9] and Beg and Shahzad [1]. This paper is a continuation of these investigations. We have obtained random fixed points of weakly inward random operators in the shell, a stochastic analogue of the results of Deimling [3].

2. Preliminaries

Let \((\Omega, A)\) be a measurable space with \(A\) as a sigma-algebra of subsets of \(\Omega\). Let \(X\) be a real Banach space; a map \(q: \Omega \to X\) is called measurable if for each open subset \(G\) of \(X\), \(q^{-1}(G) \in A\). Let \(S\) be a nonempty subset of \(X\); a map \(f: \Omega \times S \to X\) is called a random operator if for each fixed \(x \in S\), the map \(f(\cdot, x): \Omega \to X\) is measurable. A measurable map \(\xi: \Omega \to S\) is a random fixed point of the random operator \(f\) if \(f(\omega, \xi(\omega)) = \xi(\omega)\) for each \(\omega \in \Omega\).

Let \(K \subset X\) be a cone; that is, \(K\) is closed and convex such that \(\lambda K \subset K\) for all \(\lambda > 0\) and \(K \cap (-K) = \{0\}\). Denote \(\{x \in K: \|x\| \leq r\}\) by \(K_r\) and \(\{x \in K: \rho \leq \|x\| \leq r\}\) by \(K_{\rho, r}\) for
some $0 < p < r$. A map $f: K_r \to X$ is $\alpha$-condensing if $f$ is continuous and bounded and $\alpha(f(B)) < \alpha(B)$ for all $B \subset K_r$ with $\alpha(B) > 0$, where $\alpha(B) = \inf\{d > 0 : B$ can be covered by a finite number of sets of diameter $\leq d\}$. A random operator $f: \Omega \times K_r \to X$ is $\alpha$-condensing if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is $\alpha$-condensing. An operator $f: \Omega \times X \to X$ is said to be weakly inward on a closed convex subset $C$ of $X$ if for each $\omega \in \Omega$, $f(\omega, x) \in \overline{I_c(x)}$ for all $x \in C$, where $\overline{I_c(x)}$ denotes the closure of $\{x + \lambda(y - x) : \lambda \geq 0, y \in C\}$. In the case $C = K$, it simply becomes $x \in \partial K$. The latter, $x^* \in K^*$ (the dual cone), and $x^*(x) = 0$ imply that

$$x^*(f(\omega, x)) \geq 0$$

for each $\omega \in \Omega$, where $X^*$ is the dual space, $K^* = \{x^* \in X^* : x^*(x) \geq 0 \text{ on } K\}$ and $\partial K$ is the boundary of $K$. Suppose $f: \Omega \times K_r \to X$, satisfies $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ and $x^*(x) = 0$ imply that $x^*(f(\omega, x)) \geq 0$ for each $\omega \in \Omega$. Let $P_r: K_r \to K_r$ be the radial projection; that is, $P_r(x) = x$ for $\|x\| \leq r$ and $P_r(x) = \frac{rx}{\|x\|}$ for $\|x\| > r$. Then for each $\omega \in \Omega$, $f(\omega, \cdot) \circ P_r$ satisfies (1). Since $\alpha(P_r(B)) \leq \alpha(B)$ for all bounded $B \subset K$, $f(\omega, \cdot) \circ P_r$ is $\alpha$-condensing if $f$ is.

### 3. Main Results

**Theorem 1:** Let $X$ be a separable Banach space, $K \subseteq X$ a cone and $f: \Omega \times K_r \to X$ an $\alpha$-condensing random operator, such that

(i) $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ and $x^*(x) = 0$ imply that $x^*(f(\omega, x)) \geq 0$ for all $\omega \in \Omega$, and

(ii) $f(\omega, x) \neq \lambda x$ on $\|x\| = r$ and for all $\omega \in \Omega$ and $\lambda > 1$, are satisfied.

Then $f$ has a random fixed point.

**Proof:** Define a random operator $g: \Omega \times K_r \to X$ by $g(\omega, x) = (f(\omega, \cdot) \circ P_r)x$. It is clear from Deimling [3, proof of Theorem 1] that for any $\omega \in \Omega$, $g(\omega, x)$ is $\alpha$-condensing and weakly inward on $K_r$ for some $r > 0$. Further application of Xu [10, Theorem 2] or Tan and Yuan [9, Corollary 2.6] gives the desired result.

**Theorem 2:** Let $X$ be a separable Banach space, $K \subseteq X$ a cone, $f: \Omega \times K_r \to X$ an $\alpha$-condensing random operators, such that (i), (ii) (from Theorem 1) and

(iii) there exists $\rho \in (0, r)$ and $e \in K \setminus \{0\}$ such that for any $\omega \in \Omega$, $x - f(\omega, x) \neq \lambda e$ for $\|x\| = \rho$ and $\lambda > 0$

are satisfied.

Then $f$ has a random fixed point in $K_{\rho, r}$.

**Proof:** Let $P_r$ be as before and for all $\omega \in \Omega$,

$$C(\omega) = \sup\{\|f(\omega, x)\| : x \in K_r\}.$$

We use (ii) to get a “barrier” at $\|x\| = \rho$ as follows. Let $\phi_n: \Omega \times [0, r] \to [0, \infty]$ be a continuous random map such that $\phi_n(\omega, t) = 0$ for $t \geq \rho$ and $\phi_n(\omega, t) = \delta(\omega)$ for $t \leq \rho - \frac{1}{n}$ and large $n$, with a measurable map $\delta: \Omega \to (0, \infty)$ such that $\delta(\omega)\|x\| > \rho + C(\omega)$ for each $\omega$. Let $f_n(\omega, x) = (f(\omega, \cdot) \circ P_r)x + \phi_n(\omega, \|x\|)e$. Evidently, $f_n$ is $\alpha$-condensing and weakly inward random operator on $K$. By Theorem 1, there exists a measurable map $\zeta_n: \Omega \to K$ such that $\zeta_n(\omega) = f_n(\omega, \zeta_n(\omega))$ for each $\omega \in \Omega$. Fix $\omega \in \Omega$ arbitrarily. We cannot have $\|\zeta_n(\omega)\| > r$. Hence, $\zeta_n(\omega) = f(\omega, \zeta_n(\omega)) + \phi_n(\omega, \|\zeta_n(\omega)\|)e$. By the choice of $\phi_n$ we cannot have $\|\zeta_n(\omega)\| \leq \rho - \frac{1}{n}$, and we are done if $\|\zeta_n(\omega)\| \geq \rho$. So assume that $\rho - \frac{1}{n} < \|\zeta_n(\omega)\| < \rho$ for all large $n$. Since $\{\phi_n(\omega, \|\zeta_n(\omega)\|)\}$ is bounded and $f$ is $\alpha$-condensing, we have without loss of generality $\zeta_n(\omega) =$
f(ω, ζ_0(ω)) → λe and ζ_0(ω) → ζ_0(ω) with \| ζ_0(ω) \| = ρ where λ depends on ω. Hence, ζ_0(ω) = f(ω, ζ_0(ω)) + λe and therefore λ = 0 by (iii).

**Theorem 3:** Let Ω be a separable Banach space, K ⊆ X a cone such that K_1 is not compact, f: Ω × K ↠ X a compact random operator satisfying (i) and (ii) of Theorem 1 and

(iv) there exists ρ ∈ (0, r) such that f(ω, x) = f(ω, P_ρ x) \neq λx for \| x \| = ρ and λ ∈ (0, 1) and inf{\| f(ω, x) \| : \| x \| = 0} > 0, for each ω ∈ Ω.

Then f has a random fixed point in K_ρ, r.

**Proof:** Since for any ω ∈ Ω inf{\| f(ω, x) \| : \| x \| = ρ} > 0, and A = \{x ∈ K : \| x \| = 1\} is not compact, we can find e ∈ A such that \( -λe \notin (f(ω, ρA)) \) for all ω ≥ 0.

Consider a random operator f_n: Ω × K ↠ X defined by

\[ f_n(ω, x) = \begin{cases} 
(f(ω, ·) \circ P_ρ)x & \text{for } \| x \| \geq ρ \\
(f(ω, ρ \| x \| )x & \text{for } 0 < ρ_0 \leq \| x \| < ρ \\
\frac{\| x \|}{ρ_0} f(ω, ρ \| x \| )x + φ_n(ω, \| x \| )e & \text{for } \| x \| < ρ_0 
\end{cases} \]

where φ_n: Ω × [0, ρ_0] → [0, ∞) is a continuous random map, φ_n(ω, ρ_0) = 0 and φ_n(ω, t) = δ(ω) for t ≤ ρ_0(1 − \frac{1}{n}), with a measurable map δ: Ω → (0, ∞) such that

δ(ω) > ρ_0 + sup\{\| f(ω, x) \| : \| x \| = ρ\} for each ω ∈ Ω.

Since f_n is compact and a weakly inward random operator, it has a random fixed point ξ_n: Ω ↠ K_ρ as before. We cannot have ρ_0 ≤ \| ξ_n(ω) \| < ρ or \| ξ_n(ω) \| ≤ ρ_0(1 − \frac{1}{n}) by (iv) and the choice of φ_n. If, for any ω ∈ Ω, \| ξ_n(ω) \| ≥ ρ, then we are done. Now assume that there exists ω such that \( ρ_0 - \frac{1}{n} < \| ξ_n(ω) \| < ρ_0 \) for all large n. Then

\[ \| ξ_n(ω) \| = \| \frac{ρ}{\| ξ_n(ω) \|} ξ_n(ω) \| = ρ. \]

By letting n → ∞ we get \( \frac{ρ}{\| ξ_n(ω) \|} y_0 = f(ω, y_0) + λe \) with \( y_0 ∈ ρA \) and \( 0 \leq λ ≤ δ(ω) \) depending on ω, hence ρ_0 → 0 yields \(-λ_0 e ∈ f(ω, ρA)\) for some \( λ_0 > 0 \), contradicting the choice of e.

**Remark 4:** Theorem 3 does not hold if K_1 is compact. For a counterexample, see [3].

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**References**


