EXISTENCE OF ASYMPTOTIC SOLUTIONS
OF SECOND ORDER NEUTRAL DIFFERENTIAL
EQUATION WITH MULTIPLE DELAYS

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ABSTRACT

The existence of positive solutions of second order neutral differential
equation of the form

\[ x(t) - cx(t - h)'' + \sum_{i=1}^{m} p_i(t)x[g_i(t)] = 0, \quad t \geq t_0, \]

is investigated. Some sufficient conditions are given for the existence of positive
solutions with asymptotic decay of (A). Examples are presented to illustrate the
results.

Key words: Neutral Delay Differential Equation, Oscillation, Asymptotic
Positive Solution.

AMS (MOS) subject classifications: 34K05, 34K10, 34C10, 34K25.

1. Introduction

Recently, several authors [1-7] have studied the oscillatory and nonoscillatory behavior of
neutral differential equations. The main reason for this interest is that delay differential
equations play an important role in applications. For instance, in biological applications, delay
equations give better description of fluctuations in population than the ordinary ones. Also
neutral delay differential equations appear as models of electrical networks which contain lossless
transmission lines. Such networks arise, for example, in high speed computers where lossless
transmission lines are used to interconnect switching circuits.

In [8] Zhang and Yu studied the existence of positive solutions of neutral delay differential
equation (NDDE) of the form

\[ x(t) - cx(t - h)'' + p(t)x[g(t)] = 0, \quad t \geq t_0. \]  

In this paper we present some sufficient conditions for the existence of positive solutions of
second order neutral delay differential equation of the form

\[ x(t) - cx(t - h)'' + \sum_{i=1}^{m} p_i(t)x[g_i(t)] = 0, \quad t \geq t_0, \]
where \( c, h \in R_+ \), \( p_i, g_i \in C([t_0, \infty), R) \) for \( i = 1, \ldots, m \) and \( g_i(t) \to \infty \) as \( t \to \infty \) (\( i = 1, \ldots, m \)).

In order to establish the results, we need the following theorem.

**Krasnosel’skii’s fixed point theorem:** Let \( X \) be a Banach space, \( Y \) be a bounded closed convex subset of \( X \) and let \( A, B \) be mappings of \( Y \) into \( X \) such that \( Ax + By \in Y \) for every \( x, y \in Y \). If \( A \) is a strict contraction mapping and \( B \) is completely continuous then the equation

\[
Ax + Bx = x
\]

has a solution in \( Y \).

2. **Main Results**

**Theorem 1:** Assume that:

(i) \( 0 < c < 1, h > 0, p_i(t) \leq 0 \) for \( i = 1, \ldots, m \),

(ii) there exists a constant \( \alpha > 0 \) such that

\[
 ce^{oh} + \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)\exp[\alpha(t-g_i(s))]ds \leq 1.
\]

(3)

Then equation (2) has a positive solution \( x(t) \) satisfying \( x(t) \to 0 \) as \( t \to \infty \).

**Proof:** It is easy to show that if the equality in (3) holds, then equation (2) has a positive solution \( x(t) = e^{-\alpha t} \).

In the rest of the proof, we assume that there exists a number \( T > t_0 \) such that \( T - h \geq t_0 \), \( g_i(t) \leq t_0 \) for \( t \geq T \) (\( i = 1, \ldots, m \)),

\[
 \beta := ce^{oh} + \sum_{i=1}^{m} \int_{T}^{\infty} (T-s)p_i(s)\exp[\alpha(T-g_i(s))]ds < 1
\]

(4)

and condition (3) holds for \( t \geq T \).

Let \( X \) denote the Banach space of all continuous, bounded functions defined on \([t_0, \infty)\) taking values in \( R \). The space \( X \) is endowed with the supremum norm.

Let \( Y \) be the subset of \( X \) defined by

\[
 Y = \{ y \in X : 0 \leq y(t) \leq 1 \text{ for } t \geq t_0 \}.
\]

Define a mapping \( S : Y \to X \) by the formula

\[
 (Sy)(t) = (S_1y)(t) + (S_2y)(t),
\]

where

\[
 (S_1y)(t) = \begin{cases} 
 ce^{oh}y(t-h), & t \geq T \\
 (S_1y)(T) + \exp[\epsilon(T-t)] - 1, & 0 \leq t \leq T,
\end{cases}
\]

(5)

and

\[
 (S_2y)(t) = \begin{cases} 
 \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)\exp[\alpha(t-g_i(s))]y[g_i(s)]ds, & t \geq T, \\
 (S_2y)(T), & 0 \leq t \leq T,
\end{cases}
\]

(6)

and \( \epsilon = [\ln(2-\beta)]/(T-t_0) \).
It is easy to see that the integral in $S_2$ is defined whenever $y \in Y$.

Clearly, the set $Y$ is closed, bounded and convex in $X$. We shall show that for every $x, y \in Y$

$$S_1x + S_2y \in Y.$$  

(7)

Indeed, for any $x, y \in Y$, we have

$$S_1x(t) + (S_2y)(t) = ce^{\alpha h}x(t-h) + \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(x)exp[\alpha(t-g_i(s))]y[g_i(s)]ds$$

$$\leq ce^{\alpha h} + \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]ds$$

$$\leq 1 \text{ for } t \geq T$$

and

$$(S_1x)(t) + (S_2y)(t) = (S_1x)(T) + (S_2y)(T) + exp[\epsilon(T-t)] - 1$$

$$\beta + exp[\epsilon(T-t)] - 1 = \beta + exp[\epsilon(T-t_0)] - 1 = 1 \text{ for } t_0 \leq t \leq T.$$  

Obviously, $(S_1x)(t) + (S_2y)(t) \geq 0$ for $t \geq t_0$. Thus (7) is proved.

Since $0 < ce^{\alpha h} < 1$, it follows that $S_1$ is a strict contraction.

We shall show that $S_2$ is completely continuous. Indeed, from condition (3) there exists a

positive constant $M$ such that

$$|\frac{d}{dt}(S_2y)(t)| = |\sum_{i=1}^{m} \int_{t}^{\infty} p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]|$$

$$+ \alpha \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds$$

$$\leq M + \alpha \text{ for } t \geq T.$$  

Moreover, $\frac{d}{dt}(S_2y)(t) = 0$ for $t_0 \leq t \leq T$.

This implies that $S_2$ is relatively compact. On the other hand, it is easy to see that $S_2$ is

continuous and uniformly bounded, and so $S_2$ is a completely continuous mapping.

By Krasnosel'skii's fixed point theorem, $S$ has a fixed point $y \in Y$, that is

$$y(t) = \begin{cases} 
    ce^{\alpha h}y(t-h) + \sum_{i=1}^{m} \int_{t}^{\infty} (t-s)p_i(s)exp[\alpha(t-g_i(s))]y[g_i(s)]ds, t \geq T, \\
    y(T) + exp[\epsilon(T-t)] - 1, \quad t_0 \leq t \leq T.
\end{cases}$$  

(8)
Since \( y(t) \geq \exp[\epsilon(T - t)] - 1 > 0 \) for \( t_0 \leq t \leq T \), it follows that \( y(t) > 0 \) for \( t \geq t_0 \). Set
\[
x(t) = y(t)e^{-\alpha t}.
\]
(9)

Then (8) becomes
\[
x(t) = cx(t - h) + \sum_{i=1}^{m} \int_{t}^{\infty} (t - s)p_i(s)x[g_i(s)]ds, \quad t \geq T.
\]
(10)

It follows that
\[
[x(t) - cz(t - h)]'' + \sum_{i=1}^{m} p_i(t)x[g_i(t)] = 0 \quad \text{for} \quad t \geq T.
\]
It means that \( x(t) \) is a positive solution of equation (2) and \( x(t) \to 0 \) as \( t \to \infty \). The proof is complete.

**Theorem 2**: Assume that:

(i) \( c > 0 \), \( p_i(t) \geq 0 \) and \( g_i(t + h) < t \), for \( i = 1, \ldots, m \),

(ii) there exists a constant \( \alpha > 0 \) such that

\[
(1/c)e^{-\alpha h} + (1/c)\sum_{i=1}^{m} \int_{T+h}^{\infty} (s - t - h)p_i(s)\exp[\alpha(t - g_i(s))]ds \leq 1
\]
(11)

for all sufficiently large \( t \). Then equation (2) has a positive solution \( x(t) \) and \( x(t) \to 0 \) as \( t \to \infty \).

**Proof**: If the equality in (11) holds, then \( x(t) = e^{-\alpha t} \) is a solution. Now, we assume that there exists \( T > t_0 \) such that \( t + h \geq t_0 \) for \( t \geq T \),

\[
\beta = (1/c)e^{-\alpha h} + (1/c)\sum_{i=1}^{m} \int_{T+h}^{\infty} (s - T - h)p_i(s)\exp[\alpha(T - g_i(s))]ds < 1
\]
(12)
and (11) holds for \( t \geq T \).

Define the Banach space \( X \) and its subset \( Y \) as in the proof of Theorem 1.

Next, define a mapping \( S: Y \to X \) by the formula
\[
(Sy)(t) = (S_1y)(t) + (S_2y)(t),
\]
(13)
where
\[
(S_1y)(t) = \begin{cases} 
(1/c)e^{\alpha h}y(t + h), & \text{if} \quad t \geq T \\
(S_1y)(T) + \exp[\epsilon(T - t)] - 1 & \text{if} \quad t_0 \leq t \leq T 
\end{cases}
\]
and
\[
(S_2y)(t) = \begin{cases} 
(1/c)\sum_{i=1}^{m} \int_{t+h}^{\infty} (s - t - h)p_i(s)\exp[\alpha(t - g_i(s))]y[g_i(s)]ds, & \text{if} \quad t \geq T \\
(S_2y)(T), & \text{if} \quad t_0 \leq t \leq T 
\end{cases}
\]
and \( \epsilon = [\ln(2 - \beta)]/(T - t_0) \).

We can easily show that the mapping \( S \) satisfied all the conditions of Krasnosel'skii's fixed point theorem, and so \( S \) has a fixed point \( y \) in \( Y \). Clearly, \( y(t) > 0 \) for \( t \geq t_0 \) and \( x(t) = y(t)e^{-\alpha t} \) is a solution of equation (2), and so the proof is complete.

**Theorem 3**: Assume that \( c > 1 \) and
Then equation (2) has a bounded, positive solution.

**Proof:** Let $T > t_0$ be a sufficiently large number such that $T + h > t_0$, $g_i(t + h) > t_0$ ($i = 1, \ldots, m$) for $t \geq T$ and

$$
\sum_{i=1}^{m} \int_{t+h}^{\infty} s |p_i(s)| \, ds \leq \frac{c-1}{4}, \quad t \geq T.
$$

Let $X$ be the Banach space of all continuous, bounded functions defined on $[t_0, \infty)$ with the norm

$$
\|x\| = \sup_{t \geq t_0} |x(t)|.
$$

Set

$$
Y = \{x \in X : \frac{1}{2}c \leq x(t) \leq 2c \text{ for } t \geq t_0\}.
$$

Clearly, $Y$ is a bounded, closed, convex subset of $X$. Define a mapping $S: Y \to X$ by the formula

$$
(Sx)(t) = \begin{cases}
(c-1) + (1/c)x(t + h) + (1/c) \sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h)p_i(s)x[g_i(s)] \, ds, & t \geq T, \\
(Sx)(T), & t_0 \leq t \leq T
\end{cases}
$$

It is easy to show that $SY \subseteq Y$. Indeed, for any $x \in Y$ we have

$$
(Sx)(t) \leq (c-1) + (1/c)2c + (1/c) \sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h)p_i(s)2cds
$$

$$
\leq \frac{2c+1}{2} < 2c \text{ for } t \geq T
$$

and

$$
(Sx)(t) \geq (c-1) + (1/2) - 2(c-1)/4 = (1/2)c \text{ for } t \geq T.
$$

Consequently, $(1/2)c \leq (Sx)(T) \leq 2c$ for $t_0 \leq t \leq T$. Therefore, $SY \subseteq Y$.

We shall show that $S$ is a contraction. For any $x_1, x_2 \in Y$, we have

$$
| (Sx_1)(t) - (Sx_2)(t) | \leq (1/c) |x_1(t + h) - x_2(t + h)| + (1/c) \sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h) |p_i(s)| |x_1[g_i(s)] - x_2[g_i(s)]| \, ds
$$

$$
\leq (1/c) \|x_1 - x_2\| + (1/c) \|x_1 - x_2\| \sum_{i=1}^{m} \int_{t+h}^{\infty} (s-t-h) |p_i(s)| \, ds
$$

$$
\leq \|x_1 - x_2\| [(1/c) + (1/c)(c-1)/4]
$$
which implies that
\[
\| Sx_1 - Sx_2 \| = \sup_{t \geq t_0} |(Sx_1)(t) - (Sx_2)(t)|
\]
\[
= \sup_{t \geq T} |(Sx_1)(t) - (Sx_2)(t)|
\]
\[
\leq (1/4)(1 + (3/c)) \| x_1 - x_2 \| .
\]

Since \((1/4)(1 + (3/c)) < 1\), then \(S\) is a contraction. Hence, there exists a fixed point \(x \in Y\). It is easy to see that \(x(t)\) is a bounded solution of equation (2). The proof is complete. \(\square\)

3. Examples

Example 1: Consider
\[
[x(t) - cx(t - h)]'' + p_1(t)x[g_1(t)] + p_2(t)x[g_2(t)] = 0, \quad t \geq t_0,
\]
where \(c = (1/2e), \ h = 1, \ p_1(t) = p_2(t) = (-1/4)e^{-t}(1 + t)\) and \(g_1(t) = g_2(t) = \log(1 + t)\). Then \(x(t) = e^{-t} - 1\) and choosing \(\alpha = 1\), it is easy to see that
\[
ce^{\alpha h} + \sum_{i=1}^{\infty} \int_t^\infty (t - s)p_i(s)\exp[\alpha(t - g_i(s))]ds \leq 1.
\]
Hence equation (17) satisfies the hypotheses of Theorem 1. Thus this equation has a solution \(x(t) = e^{-t} - 1\) which tends to zero as \(t \to \infty\).

Example 2: Assume that
\[
c = (e^2 - 4)/(e), \ h = 1, \ p_1(t) = p_2(t) = e^{-t/2} \text{ and } g_1(t) = g_2(t) = (t/2).
\]
Then \(x(t) = e^{-t} - 1\) and choosing \(\alpha = 1\), it is easy to see that
\[
(1/c)e^{\alpha h} + (1/c)\sum_{i=1}^{\infty} \int_t^\infty (t - s)p_i(s)\exp[\alpha(t - g_i(s))]ds \leq 1.
\]
Hence equation (17) with conditions (19) satisfies the hypotheses of Theorem 2. Thus equation (17) with conditions (19) has a solution \(x(t) = e^{-t} - 1\) which tends to zero as \(t \to \infty\).

Example 3: Let
\[
p_1(t) = p_2(t) = (\sin t/t)e^{-t}, \quad t > t_0 > 0,
\]
\[
c = 1 + 4/e^2, \ h = 1,
\]
\[
g_1(t) = g_2(t) = (1/2)[t + \log[2 \sin t/(c - 1)]
\]
in equation (17). Then

\[
\sum_{i=1}^{2} \int_{t+i}^{\infty} s \cdot |p(s)| \, ds \leq \frac{(c - 1)}{4}.
\]

(22)

Therefore, equation (17) with conditions (21) satisfies the hypotheses of Theorem 3. Hence equation (17) with conditions (21) and (22) has a solution \( x(t) = e^{-t-1} \) which is a bounded positive solution.

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**References**


