A SYSTEM OF IMPULSIVE DEGENERATE NONLINEAR PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES

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(Received April, 1994; revised June, 1994)

ABSTRACT

A theorem about a system of strong impulsive degenerate nonlinear parabolic functional-differential inequalities in an arbitrary parabolic set is proved. As a consequence of the theorem, some theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem are established.

Key words: Impulsive Parabolic Problems, Diagonal Systems, Functional-Differential Inequalities, Impulsive Conditions, Uniqueness Criterion, Arbitrary Parabolic Sets.

AMS (MOS) subject classifications: 35K65, 35R10, 35K85, 35K60, 35K99.

1. Introduction

In this paper we prove a theorem about strong inequalities for the following diagonal system of degenerate nonlinear parabolic functional-differential inequalities

\[ F_i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x),u) > F_i(t,x,v(t,x),v^i_t(t,x),v^i_x(t,x),v^i_{xx}(t,x),v) \quad (i = 1, \ldots, m), \]

where \((t,x) \in D \setminus \bigcup_{j=1}^{s} \{t_j \times \mathbb{R}^n\}, \quad t_0 < t_1 < \ldots < t_s < t_0 + T\) and \(D\) is a relatively arbitrary set more general than the cylindrical domain \((t_0, t_0 + T) \times \Omega_0 \subset \mathbb{R}^{n+1}\). In the expressions

\[ F_i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),w^i_{xx}(t,x),w) \quad (i = 1, \ldots, m) \]

the symbol \(w\) denotes a function

\[ w: \tilde{D} \ni (t,x) \mapsto w(t,x) = (w^1(t,x), \ldots, w^m(t,x)) \in \mathbb{R}^m, \]

where \(\tilde{D}\) is an arbitrary set such that \(\tilde{D} \cap [(t_0, t_0 + T) \times \mathbb{R}^n] \subset \tilde{D} \subset (-\infty, t_0 + T) \times \mathbb{R}^n\),

\[ w^i_x(t,x): = \text{grad}_x w^i(t,x) \quad (i = 1, \ldots, m) \quad \text{and} \quad w^i_{xx}(t,x): = \left[ \frac{\partial^2 w^i(t,x)}{\partial x_j \partial x_k} \right]_{n \times n} \quad (i = 1, \ldots, m). \]
assume that the limits $w(t_j^-, x)$, $w(t_j^+, x)$ ($j = 1, \ldots, s$) exist for all admissible $x \in \mathbb{R}^n$, they are finite, all different and $w(t_j, x) = w(t_j^+, x)$ ($j = 1, \ldots, s$) for all admissible $x \in \mathbb{R}^n$.

System (1.1) is studied together with impulsive and boundary inequalities. The impulsive inequalities are of the form

$$u(t_j, x) - u(t_j^-, x) \leq v(t_j, x) - v(t_j^-, x) \quad (j = 1, \ldots, s).$$

As a consequence of the theorem about the strong inequalities for system (1.1), we establish theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem.

The results obtained in the paper are direct generalizations of those given by the author in [2]. To prove the results of this paper, theorems of [2] are used. The paper is a continuation of author’s publication [3] about impulsive parabolic problems. The impulsive conditions in the present paper are quite different from those considered in [3]. They are similar to the impulsive conditions used by Bainov, Kamont and Minchev in [1].

2. Preliminaries

We use the notation: $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$. For any vectors $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_m) \in \mathbb{R}^m$ we write $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ ($i = 1, \ldots, m$).

By $\Omega$ we denote an arbitrary open subset of $(t_0, t_0 + T) \times \mathbb{R}^m$, where $t_0 \in \mathbb{R}$ and $T \in \mathbb{R}_+ \setminus \{0\}$, such that the projection of $\Omega$ on the $t$-axis is the interval $(t_0, t_0 + T)$.

Next, by $D$ we denote the subset of the set $\overline{\Omega} \cap [(t_0, t_0 + T) \times \mathbb{R}^m]$ satisfying the condition that for any $(\tilde{t}, \tilde{x}) \in D$ there exists a number $\rho > 0$ such that

$$\{(t, x): (t - \tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < \rho, \quad t < \tilde{t} \} \subset \Omega.$$

It is clear that $\Omega \subset D$.

We define the sets

$$S_{t_i} = \{x \in \mathbb{R}^n: (t, x) \in \overline{D} \} \quad \text{for} \quad t \in [t_0, t_0 + T]$$

and

$$\sigma_{t_i} = \overline{D} \cap \{(t) \times \mathbb{R}^n \} \quad \text{for} \quad t \in [t_0, t_0 + T].$$

By $s$ we denote a fixed number belonging to $\mathbb{N}$.

Let $t_1, t_2, \ldots, t_s$ be arbitrary fixed real numbers such that

$$t_0 < t_1 < \ldots < t_s < t_0 + T.$$

We introduce the following sets:

$$D_j = D \cap [(t_j, t_{j+1}) \times \mathbb{R}^n] \quad (j = 0, 1, \ldots, s - 1),$$
Let $\sigma_j$ be an arbitrary set such that $\sigma_j \subseteq (-\infty, t_0 + T) \times \mathbb{R}^n$, $t_0 + T \leq \infty$.

By $\Sigma$ we denote the part of $\partial D \setminus (\sigma_{t_0} \cup \sigma_\Delta \cup \sigma_{t_0 + T})$ disjoint with $D$.

**Assumption (A):** For each $i \in \{1, \ldots, m\}$ let $\Sigma_i$ be a subset (possibly empty) of $\Sigma$ and let, for each $(t, x) \in \Sigma_i$, $\ell_i(t, x)$ be a direction. We assume that $\ell_i$ is orthogonal to the $t$-axis and some open segment, with one extremity at $(t, x)$, of the ray with origin at $(t, x)$ in the direction of $\ell_i$ is contained in $D$.

Given a subset $E$ of $\sigma_{t_0} \cup \sigma_\Delta \cup \Sigma$ [of $\sigma_\Delta$] and a function $\omega: D \rightarrow \mathbb{R}$, we say that $\omega$ has **finite right-hand [left-hand]** sided limits in $E \cup \{\infty\}$ if for every $(\bar{t}, \bar{x}) \in E$ and every $\bar{t} \in P(E)$, and for each sequence $(t', x') \in D$, such that $t' \to \bar{t}$ [as $t' \to \bar{t}$] and $(t', x') \to (\bar{t}, \bar{x})$ or $|x'| \to \infty$, the limit $\lim \omega(t', x')$ is finite; here $P(E)$ is the projection of $E$ on the $t$-axis.

Obviously, this limit does not depend on the choice of the sequence $(t', x')$ and it will be denoted by $\omega_{\bar{t}}(\bar{x})$ and $\omega_{\bar{t}, \infty}(\bar{x})$, respectively.

Let $E$ be a subset of $\sigma_{t_0} \cup \sigma_\Delta \cup \Sigma$. If for $\bar{t} \in P(E)$ there is a sequence $(t', x') \in D$ such that $t' \to \bar{t}$, $t' \to \bar{t}$ and $|x'| \to \infty$ then we denote by $(\bar{t}, \infty)$ the class of all such sequences. By a function $\varphi: E \cup \{\infty\} \to \mathbb{R}$ we mean a function defined for $(t, x) \in E$ and $(t, \infty)$ with $t \in P(E)$.

By $PC_{m}(\tilde{D})$ we denote the space of mappings

$$w: \tilde{D} \ni (t, x) \mapsto w(t, x) = (w^1(t, x), \ldots, w^m(t, x)) \in \mathbb{R}^m,$$

such that, for every $i \in \{1, \ldots, m\}$, $w^i$ is continuous in $(D \cup \Sigma_i) \setminus \sigma_\Delta$, has finite right-hand sided limits $w^i(t^+, x)$, $w^i(t^+, \infty)$ in $\sigma_{t_0} \cup \sigma_\Delta \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}$, has finite left-hand sided limits $w^i(t^-, x)$, $w^i(t^-, \infty)$ in $\sigma_{t_0} \cup \{\infty\}$, and $w^i(t, x) = w^i(t^+, x)$ for $(t, x) \in \sigma_{t_0} \cup \sigma_\Delta \cup (\Sigma \setminus \Sigma_i)$ and $w^i(t, \infty) = w^i(t^+, \infty)$ for $t \in P(\sigma_{t_0} \cup \sigma_\Delta \cup (\Sigma \setminus \Sigma_i))$.

For $w, \bar{w} \in PC_{m}(\tilde{D})$ and for every fixed $t < t_0 + T$, we write $w \preceq \bar{w}$ if $w(t, x) \leq \bar{w}(t, x)$ for $(t, x) \in \tilde{D}$, $t \leq t$ ($i = 1, \ldots, m$). Given the sets $\Sigma_i$ ($i = 1, \ldots, m$) and the directions $\ell_i$ ($i = 1, \ldots, m$) satisfying Assumption (A), a function $w \in PC_{m}(\tilde{D})$ is said to **belong to** $PC_{m, \Sigma}(\tilde{D})$ if $w_i, w_{tx}, w_{xx}$ ($i = 1, \ldots, m$) are continuous in $D_\Delta$ and the derivatives $\frac{d w^i}{d \ell_i}$ ($i = 1, \ldots, m$) are finite on $\Sigma_i$ ($i = 1, \ldots, m$), respectively.

By $M_{n \times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$. For each $i \in \{1, \ldots, m\}$ by $F_i$ we define the mapping

$$F_i: D_\Delta \times \mathbb{R}^m \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \times PC_{m, \Sigma}(\tilde{D}) \ni (t, x, z, p, q, r, w) \mapsto F_i(t, x, z, p, q, r, w) \in \mathbb{R},$$

where $q = (q_1, \ldots, q_n)$ and $r = [r_{jk}]_{n \times n}$.

We use the notation

$$F_i(t, x, w) = F_i(t, x, w(t, x), w_x^i(t, x), w_{xx}^i(t, x), w^i(t, x), w^i(t, x), w^i(t, x)) \quad (i = 1, \ldots, m)$$
for all \((t,x) \in D_0\) and \(w \in PC^{1,2}_{m, \Sigma} (\tilde{D})\).

By \(Z\) we denote a fixed subset of \(PC^{1,2}_{m, \Sigma} (\tilde{D})\). Functions \(u\) and \(v\) belonging to \(Z\) are called solutions of the system

\[
F_i[t, x, u] > F_i[t, x, v] \quad (i = 1, \ldots, m)
\]

in \(D_0\), if they satisfy (2.1) for all \((t,x) \in D_0\).

The functions \(F_i\) \((i = 1, \ldots, m)\) are said to be parabolic with respect to \(w \in PC^{1,2}_{m, \Sigma} (\tilde{D})\) in \(D_0\) if for every \(r = [r_{jk}], \tilde{r} = [\tilde{r}_{jk}] \in M_{n \times n}(\mathbb{R})\) and \((t,x) \in D_0\) the following implications hold:

\[
r \leq \tilde{r} \Rightarrow F_i(t, x, u(t,x), u^i(t,x), \tilde{r}, u) \leq F_i(t, x, u(t,x), u^i(t,x), r, u) \quad (i = 1, \ldots, m),
\]

where \(r \leq \tilde{r}\) means that the inequality \(\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk})\lambda_j \lambda_k \leq 0\) is satisfied for each \((\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\).

3. Theorem about Impulsive Functional-Differential Inequalities

Theorem 3.1. Assume that:

1. The functions \(F_i\) \((i = 1, \ldots, m)\) are weakly increasing with respect to \(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m\) \((i = 1, \ldots, m)\), respectively. Moreover, \(F(i = 1, \ldots, m)\) are weakly increasing with respect to \(w\) in the sense of the relation \(\leq\) for all \(t \in (t_0, t_0 + T)\) and

\[
F_i(t, x, z, p, q, r, w) \geq F_i(t, x, z, \tilde{p}, q, r, w) \quad (i = 1, \ldots, m)
\]

for all \((t,x) \in D_0, z \in \mathbb{R}^m, p < \tilde{p}, q \in \mathbb{R}^n, r \in M_{n \times n}(\mathbb{R}), w \in Z\).

2. For the given sets \(\Sigma_i(i = 1, \ldots, m)\) and the directions \(\ell_i\) \((i = 1, \ldots, m)\) satisfying Assumption (A), for the given functions \(a_i: \Sigma_i \rightarrow \mathbb{R}_+ (i = 1, \ldots, m)\) and for the given functions \(\phi_i: \Sigma_i \times \mathbb{R}_+ \rightarrow \mathbb{R}(i = 1, \ldots, m)\) of the variables \((t,x,\xi)\) and weakly increasing with respect to \(\xi\), functions \(u\) and \(v\) belonging to \(Z\) satisfy the inequalities

\[
u(t,x) < v(t,x) \quad \text{for} \quad (t,x) \in \tilde{D} \setminus \tilde{D},
\]

\[
u^i(t,x) < v^i(t,x) \quad \text{for} \quad (t,x) \in \sigma_{t_0} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\} \quad (i = 1, \ldots, m),
\]

\[
u(t,x) - \nu(t_-, x) < v(t,x) - v(t_-, x) \quad \text{for} \quad (t,x) \in \sigma_0,
\]

\[
\phi_i(t,x, u^i(t,x)) - \phi_i(t,x, v^i(t,x)) < a_i(t,x) \left[\frac{du^i(t,x) - v^i(t,x)}{d\ell_i}\right] \quad (i = 1, \ldots, m)
\]

for \((t,x) \in \Sigma_i\),

and the condition

\[
u^i(t,x) \neq v^i(t,x) \quad \text{for} \quad (t,x) \in \Sigma_i \quad (i = 1, \ldots, m).
\]
3. \( F_i \ (i = 1, \ldots, m) \) are parabolic with respect to \( u \) in \( D_\ast \) and \( u, v \) are solutions of system (2.1) in \( D_\ast \).

Then,
\[
u(t, x) < v(t, x) \quad \text{for} \quad (t, x) \in \tilde{D}.
\]

**Proof.** To prove Theorem 3.1 consider the following problem:

\[
\begin{align*}
F_i[t, x, u] &> F_i[t, x, v] \quad \text{for} \quad (t, x) \in D_0 \quad (i = 1, \ldots, m), \\
u(t, x) &< v(t, x) \quad \text{for} \quad (t, x) \in (\tilde{D} \setminus \bar{D}) \cap ((-\infty, t_1) \times \mathbb{R}^n), \\
u_i(t, x) &< v_i(t, x) \quad \text{for} \quad (t, x) \in \left[\sigma_{t_0} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}\right] \\
\cap \left[\left(t_0, t_1\right] \times \mathbb{R}^n\right] \quad (i = 1, \ldots, m), \\
\phi_i(t, x, u_i(t, x)) - \phi_i(t, x, v_i(t, x)) &< a_i(t, x) \frac{d[u_i(t, x) - v_i(t, x)]}{d\ell_i} \\
\text{for} \quad (t, x) \in \Sigma_i \cap \left[\left(t_0, t_1\right] \times \mathbb{R}^n\right] \quad (i = 1, \ldots, m).
\end{align*}
\]

According to the assumptions of Theorem 3.1 corresponding to problem (3.7), by Theorem 2.1 from [2] applied to set \( D_0 \), we obtain the inequality
\[
u(t, x) < v(t, x) \quad \text{for} \quad (t, x) \in D_0.
\]

By (3.8) and by the fact that \( u, v \in PC_m(\tilde{D}) \),
\[
u(t_-, x) \leq v(t_-, x) \quad \text{for} \quad (t, x) \in \sigma_{t_1}.
\]

From (3.3) and (3.9), we have
\[
u(t, x) < v(t, x) \quad \text{for} \quad (t, x) \in \sigma_{t_1}.
\]

Inequalities (3.1), (3.8), (3.2) and (3.10) imply that
\[
u_i(t, x) < v_i(t, x) \quad \text{for} \quad (t, x) \in (\tilde{D} \setminus \Sigma_i) \cap \left[(-\infty, t_1] \times \mathbb{R}^n\right] \quad (i = 1, \ldots, m).
\]

By (3.11), (3.5) and the fact that \( u_i(i = 1, \ldots, m) \) is continuous in \( \Sigma_i \ (i = 1, \ldots, m) \), we get
\[
u(t, x) < v(t, x) \quad \text{for} \quad (t, x) \in \tilde{D} \cap \left[(-\infty, t_1] \times \mathbb{R}^n\right].
\]

Now, set the following problem:
\[ F_i[t, x, u] > F_i[t, x, v] \text{ for } (t, x) \in D_1 \quad (i = 1, \ldots, m), \]
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \left[ \tilde{D} \cap \left( -\infty, t_2 \right] \times \mathbb{R}^n \right] \setminus D_1, \]
\[ u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in \left[ \sigma_{t_1} \cup \left( \Sigma \setminus \Sigma_i \cup \{\infty\} \right) \right] \cap \left[ [t_1, t_2] \times \mathbb{R}^n \right] \quad (i = 1, \ldots, m), \]
\[ \phi_i(t, x, u^i(t, x)) - \phi_i(t, x, v^i(t, x)) < a_i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{dt}, \]
\[ \text{for } (t, x) \in \Sigma_i \cap [2, t_2] \times \mathbb{R}^n \quad (i = 1, \ldots, m). \]

According to the assumptions of Theorem 3.1 corresponding to problem (3.13), by Theorem 2.1 from [2] applied to set \( D_1 \), we arrive at the inequality
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in D_1. \] (3.14)

By (3.14) and by the fact that \( u, v \in PC_m(\tilde{D}) \),
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_2}^r. \] (3.15)

From (3.3) and (3.15), we have
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_2}^r. \] (3.16)

Inequalities (3.1), (3.12), (3.14), (3.2) and (3.16) imply that
\[ u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in \left[ \tilde{D} \cap \left( -\infty, t_2 \right] \times \mathbb{R}^n \right] \setminus \left[ \Sigma_i \cap [t_1, t_2] \times \mathbb{R}^n \right]. \] (3.17)

By (3.17), (3.5) and the fact that \( u^i \quad (i = 1, \ldots, m) \) is continuous in \( \Sigma_i \quad (i = 1, \ldots, m) \), we get
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \cap \left( -\infty, t_2 \right] \times \mathbb{R}^n. \] (3.18)

Repeating the above procedure \( s - 2 \) times, we obtain
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_s}^r \] (3.19)
and
\[ u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \cap \left( -\infty, t_s \right] \times \mathbb{R}^n. \] (3.20)

Finally, consider the problem
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\[
F_i(t, x, u) > F_i(t, x, v) \quad \text{for} \quad (t, x) \in D_s \quad (i = 1, \ldots, m),
\]
\[
u(t, x) < \nu(t, x) \quad \text{for} \quad (t, x) \in \tilde{D} \setminus D_s,
\]
\[
u'(t, x) < \nu'(t, x) \quad \text{for} \quad (t, x) \in \left[\sigma_{t_s} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}\right] 
\cap \left[\left(\Sigma_i \cap [t_s, t_0 + T) \times \mathbb{R}^n\right) \cap \{\infty\}\right],
\]
\[
\phi_i(t, x, u'(t, x)) - \phi_i(t, x, v'(t, x)) < a_i(t, x) \frac{d[u'(t, x) - v'(t, x)]}{d\ell_i}
\]
\[
\text{for} \quad (t, x) \in \Sigma_i \cap \left[(t_s, t_0 + T) \times \mathbb{R}^n\right] \quad (i = 1, \ldots, m).
\]

According to the assumptions of Theorem 3.1 corresponding to problem (3.21), by Theorem 2.1 from [2] applied to set $D_s$, we get the inequality
\[
u(t, x) < \nu(t, x) \quad \text{for} \quad (t, x) \in D_s.
\]

Inequalities (3.1), (3.20), (3.22), (3.2) and (3.19) imply that
\[
u'(t, x) < \nu'(t, x) \quad \text{for} \quad (t, x) \in \tilde{D} \setminus \left[\Sigma_i \cap \left[(t_s, t_0 + T) \times \mathbb{R}^n\right]\right] \quad (i = 1, \ldots, m).
\]

By (3.23), (3.5) and the fact that $u'(i = 1, \ldots, m)$ is continuous in $\Sigma_i$ $(i = 1, \ldots, m)$, we have
\[
u(t, x) < \nu(t, x) \quad \text{for} \quad (t, x) \in \tilde{D}.
\]

4. Theorems about Impulsive Differential Inequalities

From the proof of Theorem 3.1 it is easy to see that the following theorem is true:

**Theorem 4.1.** Assume that:
1. $\tilde{D} = D$ and the functions
\[
G_i: D_s \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \ni (t, x, z, p, q, r) \rightarrow G_i(t, x, z, p, q, r) \in \mathbb{R}
\]
are weakly increasing with respect to $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ $(i = 1, \ldots, m)$, respectively, and
\[
G_i(t, x, z, p, q, r) \geq G_i(t, x, z, \tilde{p}, q, r) \quad (i = 1, \ldots, m)
\]
for all $(t, x) \in D_s$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$.
2. For the given sets $\Sigma_i(i = 1, \ldots, m)$ and the directions $\ell_i(i = 1, \ldots, m)$ satisfying Assumption (A), for the given functions $a_i: \Sigma_i \rightarrow \mathbb{R}_+$ $(i = 1, \ldots, m)$ and for the given functions $\phi_i: \Sigma_i \times \mathbb{R} \rightarrow \mathbb{R}$ $(i = 1, \ldots, m)$ of the variables $(t, x, \xi)$ and weakly increasing with respect to $\xi$, functions $u$ and $v$ belonging to $Z \subset PC_{\min, \Sigma}^2(\tilde{D})$ satisfy inequalities (3.2)-(3.4).
3. $G_i(i = 1, \ldots, m)$ are parabolic with respect to $u$ in $D_s$, and $u, v$ are solutions of the system
\[
G_i(t, x, u(t, x), u_x(t, x), u_{xx}(t, x))
\]
Then
(i) \( u^i(t,x) < v^i(t,x) \) for \((t,x) \in (\bar{D} \setminus \Sigma_i) \cap ([t_0, t_0 + T) \times \mathbb{R}^n) \) \((i = 1, \ldots, m)\)
and
(ii) \( u^i(t,x) \leq v^i(t,x) \) for \((t,x) \in \Sigma_i \) \((i = 1, \ldots, m)\).

Moreover,
(iii) \( u(t,x) < v(t,x) \) for \((t,x) \in \bar{D} \cap ([t_0, t_0 + T) \times \mathbb{R}^n) \)
if (3.5) holds.

As a consequence of Theorem 4.1, we obtain the following theorem:

**Theorem 4.2.** Assume that:

1. \( \bar{D} = D \) and the function
   \[ G: D_* \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \ni (t,x,z,p,q,r) \rightarrow G(t,x,z,p,q,r) \in \mathbb{R} \]
   is weakly decreasing with respect to \( z \) and \( p \) in \( D_* \).

2. For the given set \( \bar{\Sigma} \subseteq \Sigma \) and the direction \( \ell \) satisfying Assumption (A), for the given function \( \phi: \bar{\Sigma} \rightarrow \mathbb{R}_+ \) and for the given function \( \phi: \bar{\Sigma} \times \mathbb{R} \rightarrow \mathbb{R} \) of the variables \((t,x,\xi)\) and strictly increasing with respect to \( \xi \), functions \( u \) and \( v \) belonging to \( Z \subseteq PC_{1,\Sigma}^1(D) \) satisfy the inequalities
   \[ u(t,x) \leq v(t,x) \] for \((t,x) \in \sigma_{t_0} \cup (\Sigma \setminus \bar{\Sigma}) \cup \{\infty\}, \]
   \[ u(t,x) - u(t^-,x) \leq v(t,x) - v(t^-,x) \] for \((t,x) \in \sigma_{*} \)
and
   \[ \phi(t,x,u(t,x)) - \phi(t,x,v(t,x)) \leq \alpha(t,x) \frac{d[u(t,x) - v(t,x)]}{d\ell} \] for \((t,x) \in \bar{\Sigma}. \)

3. \( G \) is parabolic with respect to \( u \) in \( D_* \) and \( u,v \) are solutions of the inequality
   \[ G(t,x,u(t,x),u_t(t,x),u_x(t,x),u_{xx}(t,x)) \]
   \[ > G(t,x,v(t,x),v_t(t,x),v_x(t,x),v_{xx}(t,x)) \] \((4.1)\)
in \( D_* \).

Then
\[ u(t,x) \leq v(t,x) \] for \((t,x) \in D_* \). \((4.2)\)

**Proof.** Let \( \epsilon > 0 \) and let
\[
\psi^\epsilon(t,x) := \left\{ \begin{array}{ll}
\psi(t,x) + \epsilon & \text{for } (t,x) \in \bar{D}_0 \setminus \sigma_{t_1}, \\
v(t,x) + 2\epsilon & \text{for } (t,x) \in \bar{D}_1 \setminus \sigma_{t_2}, \\
\ldots \ldots \\
v(t,x) + s\epsilon & \text{for } (t,x) \in \bar{D}_{s-1} \setminus \sigma_{t_s}, \\
v(t,x) + (s+1)\epsilon & \text{for } (t,x) \in \bar{D}_s \setminus \sigma_{t_0 + T}.
\end{array} \right.
\]
By (4.1), (4.3) and by the fact that \( G \) is weakly decreasing with respect to \( z \), we obtain

\[
G(t, x, u(t, x), u_t(t, x), u_x(t, x), u_{xx}(t, x)) \\
- G(t, x, v^\epsilon(t, x), v_t^\epsilon(t, x), v_x^\epsilon(t, x), v_{xx}^\epsilon(t, x)) \\
> G(t, x, v(t, x), v_t(t, x), v_x(t, x), v_{xx}(t, x)) \\
- G(t, x, v^\epsilon(t, x), v_t^\epsilon(t, x), v_x^\epsilon(t, x), v_{xx}^\epsilon(t, x)) \geq 0 \text{ for } (t, x) \in D_*.
\]

Moreover, from assumption 2 of Theorem 4.2 and from (4.3) it follows that

\[
u(t, x) < v^\epsilon(t, x) \text{ for } (t, x) \in \sigma_{t_0} \cup (\Sigma \setminus \bar{\Sigma}) \cup \{\infty\},
\]

\[
\phi(t, x, u(t, x)) - \phi(t, x, v^\epsilon(t, x)) \\
< \phi(t, x, u(t, x)) - \phi(t, x, v(t, x)) \\
\leq a(t, x) \frac{d[u(t, x) - v(t, x)]}{d\ell} \\
= a(t, x) \frac{d[u(t, x) - v^\epsilon(t, x)]}{d\ell} \text{ for } (t, x) \in \bar{\Sigma}
\]

and

\[
u(t_j, x) - u(t_j^-, x) \leq v(t_j, x) - v(t_j^-, x) \\
< [v(t_j, x) + (j + 1)\epsilon] - [v(t_j^-, x) + j\epsilon] \\
= v^\epsilon(t_j, x) - v^\epsilon(t_j^-, x) \text{ for } x \in S_{t_j} \ (j = 1, 2, \ldots, s).
\]

Then we have the inequality

\[
u(t, x) < v^\epsilon(t, x) \text{ for } (t, x) \in D_*
\]

because functions \( u \) and \( v^\epsilon \) satisfy all the assumptions of Theorem 4.1. Hence (4.2) holds.

**Remark 4.1.** From the proof of Theorem 4.2 it is easy to see that if function \( G \) from Theorem 4.2 is strictly decreasing with respect to \( z \) and weakly decreasing with respect to \( p \) in \( D_* \) then Theorem 4.2 is true if strong inequality (4.1) is replaced by the weak inequality

\[
G(t, x, u(t, x), u_t(t, x), u_x(t, x), u_{xx}(t, x)) \\
\geq G(t, x, v(t, x), v_t(t, x), v_x(t, x), v_{xx}(t, x)), \ (t, x) \in D_*.
\]

Theorem 4.2 and Remark 4.1 imply the following theorem about the uniqueness of a classical solution of a mixed impulsive parabolic differential problem:
Theorem 4.3. Assume that:

1. $\bar{D} = \bar{D}$ and the function $G$ from Theorem 4.2 is strictly decreasing with respect to $z$ and weakly decreasing with respect to $p$ in $D_\ast$.

2. The set $\tilde{\Sigma} \subset \Sigma$ and the direction $\ell$ satisfy Assumption (A), $a: \tilde{\Sigma} \to \mathbb{R}_+$ is a given function, the function $\phi: \tilde{\Sigma} \times \mathbb{R} \to \mathbb{R}$ of the variables $(t, x, \xi)$ is strictly increasing with respect to $\xi$, and $f: \sigma_0 \cup (\Sigma \setminus \tilde{\Sigma}) \cup \{\infty\} \to \mathbb{R}$, $g: \sigma_\ast \to \mathbb{R}$, $h: \tilde{\Sigma} \to \mathbb{R}$ are given functions.

Then in the class of all functions $w$ belonging to $PC_1^1,2(D)$ and such that function $G$ is parabolic with respect to $w$ in $D_\ast$ there exists at most one function satisfying the following mixed impulsive parabolic differential problem:

$$G(t, x, w(t, x), w_t(t, x), w_x(t, x), w_{xx}(t, x)) = 0, \quad (t, x) \in D_\ast,$$

$$w(t, x) = f(t, x), \quad (t, x) \in \sigma_0 \cup (\Sigma \setminus \tilde{\Sigma}) \cup \{\infty\},$$

$$w(t, x) - w(t^-, x) = g(t, x), \quad (t, x) \in \sigma_\ast,$$

$$g(t, x, w(t, x)) - a(t, x) \frac{dw(t, x)}{d\ell} = h(t, x), \quad (t, x) \in \tilde{\Sigma}.$$ 

5. Remarks

Remark 5.1. Since the functions $F_i \ (i = 1, \ldots, m)$ from Theorem 3.1 are weakly decreasing with respect to $p$ then these functions may be, particularly, defined by the following formulae:

$$F_i(t, x, z, p, q, r, w) = f_i(t, x, z, q, r, w) - c_i(t, x)p \ \ (i = 1, \ldots, m),$$

where $(t, x) \in D_\ast$, $z \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z$, and $c_i(t, x) \geq 0 \ (i = 1, \ldots, m)$ for $(t, x) \in D_\ast$.

The same remarks are true for functions $G_i(i = 1, \ldots, m)$ and $G$ from Theorems 4.1-4.3.

Therefore, the degenerate parabolic problems from this paper are more general than the parabolic problems, in the normal form with respect to $p$, corresponding to the considered degenerate parabolic problems.

Remark 5.2. Theorems 4.2 and 4.3 are formulated only for the differential parabolic problems and for $m = 1$ because assuming, simultaneously, that $F_i(i = 1, \ldots, m)$ from Theorem 3.1 are weakly increasing with respect to $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$, $w$ and weakly decreasing with respect to $z_1, \ldots, z_n$, $w$ we can consider only the differential problems, where $m = 1$.

References

