NOTE ON THE INEQUALITIES OF J. KAZDAN

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ABSTRACT

In this note, we prove the Kazdan's inequalities without using what is called the Heisenberg uncertainty principle. Instead we prove it using Garofalo-Lin inequality among other things.

Key words: Heisenberg uncertainty principle, unique continuation theorem, Garofalo-Lin inequality, Schwarz inequality, Poincare inequality.

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1. INTRODUCTION

In [4], J. Kazdan has shown strong unique continuation theorem (Theorem 1.8 of [4]) whose proof is mainly based on his main lemma (Lemma 2.4 of [4]). Several analytic as well as geometric inequalities were used to prove the main lemma. Among them are the following inequalities:

There exist constants $C_1, C_2, C_3, C_4, C_5$ and $r_0$ such that for all $r \in (0, r_0)$

$$|I_j(r)| \leq C_j f(r)(H(r) + D(r)) \quad (j = 1, 2)$$

$$\frac{1}{r^{n-2}} \int_{\partial B_r} |\nabla u|^2 dS \leq r B(r) + C_3 H(r) + D(r)$$

$$|I_3(r)| \leq C_4 f(r)(H(r) + D(r) + \sqrt{r H(r) B(r)})$$

$$|I_4(r)| \leq C_5 f(r)(H(r) + D(r) + r B(r)).$$
Here $f(r), I_1(r), I_2, I_3(r), H(r), D(r), B(r)$ are defined as follows: let $f$ be a smooth increasing function with $f(0) = 0$ satisfying $\int_0^r \frac{f(r)}{r} dr < \infty$ and let $u$ satisfy for $n \geq 3$ the differential inequality with $a$ and $b$ constants:

$$| \Delta u(x) | \leq \frac{af(r)}{r^2} | u(x) | + \frac{bf(r)}{r^2} | \nabla u(x) |$$

(5)

$$I_1(r) = \frac{1}{r^{n-2}} \int_{B_r} u \Delta u dV$$

(6)

$$I_2(r) = \frac{2}{r^{n-2}} \int_{B_r} \rho u \Delta u dV$$

(7)

$$I_3(r) = \frac{1}{r^{n-3}} \int_{\partial B_r} u \Delta u dS$$

(8)

$$H(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} | u |^2 dS$$

(9)

$$D(r) = \frac{1}{r^{n-2}} \int_{B_r} | \nabla u |^2 dV$$

(10)

$$B(r) = \frac{2}{r^{n-2}} \int_{\partial B_r} u^2 dS.$$  

(11)

In his proof of inequalities (1) - (2), Kazdan relies on what is called the Heisenberg uncertainty principle (see [2], [3] & [4]):

$$\int_{B_r} \frac{w^2}{\rho^2} dV \leq C \int_{\partial B_r} w^2 dS + \tilde{C} \int_{B_r} | \nabla w |^2 dV, \ n \geq 3$$  

(12)

$$\int_{B_r} \frac{2}{\rho} | \frac{w}{\rho} \nabla w | dV \leq C' \int_{\partial B_r} w^2 dS + \tilde{C}' \int_{B_r} | \nabla w |^2 dV, \ n \geq 3$$  

(13)

where $C$ and $\tilde{C}$ are dimensional constants. Inequality (13) is an easy consequence of (12). Indeed a straightforward computation shows that $C' = \frac{C}{\lambda^2}, \tilde{C}' = \lambda(\frac{1}{\lambda^2} + \tilde{C})$ for any $\lambda > 0$. 

Since there is nothing to comment on the proofs of inequalities (3) and (4), we prove inequalities (1,5) and (2) without using the Heisenberg uncertainty principle (12)-(13). Instead we use the following lemma which is the Garofalo-Lin inequality (see 4.11 of [2]) applied to the operator $L$ where

$$Lu = -\Delta u + b(x).\nabla u + V(x)u = 0. \quad (14)$$

Here $b(x)$ and $V$ are majorized by with constants $a$ and $b$:

$$|b(x)| \leq \frac{bf(r)}{r}, \quad |V(x)| \leq \frac{af(r)}{r^2}. \quad (15)$$

**Lemma:** Let $u \in W^{1,2}_{loc}$ satisfy equation (14). Then there exists a small constant $r_0 \in (0,1)$ depending on $n, b, V$ and $u$ such that for all $r \in (0, r_0)$

$$r \int_{\partial B_r} u^2 dS \geq \int_{B_r} u^2 dV. \quad (16)$$

**Proof:** First observe that

$$\int_{B_r} u(b(x),u)(r^2 - |x|^2)dV$$

$$\leq \int_{B_r} |u| |b(x)| |\nabla u|(r^2 - |x|^2)dV$$

$$\leq \|b\|_{L^\infty}(\int_{B_r} u^2(r^2 - |x|^2)dV)^{1/2}(\int_{B_r} |\nabla u|^2(r^2 - |x|^2)dV)^{1/2}$$

(Schwarz inequality)

$$\leq \|b\|_{L^\infty r_0^2}(\int_{B_r} u^2 dV)^{1/2}(\int_{B_r} |\nabla u|^2 dV)^{1/2}$$

$$\leq C \|b\|_{L^\infty r_0^2}\int_{B_r} |\nabla u|^2 dV \quad (Poincare inequality)$$

where $C$ is a dimensional constant. Consequently we obtain

$$\int_{B_r} u(b(x),\nabla u)(r^2 - |x|^2 dV \geq -C \|b\|_{L^\infty r_0^2}\int_{B_r} |\nabla u|^2 dV. \quad (17)$$

Choose $r_0$ so small that

$$r_0^2 \leq 1/(C \|b\|_{L^\infty}\int_{B_r} |\nabla u|^2 dV \int_{B_r} u^2 dV). \quad (18)$$
Inequalities (17)-(18) then reveal that
\[ \int_{B_r} u(b(x) \cdot \nabla u)(r^2 - |x|^2) dV \geq - \int_{B_r} u^2 dV. \] (19)

Secondly we have
\[ \int_{B_r} V u^2 (r^2 - |x|^2) dV \geq - \| V \|_{L^\infty} r_0^2 \int_{B_r} u^2 dV. \] (20)

Choose \( r_0 \) such that
\[ r_0^2 \leq (n-2)/\| V \|_{L^\infty}. \] (21)

Inequalities (20)-(21) then show that
\[ \int_{B_r} V u^2 (r^2 - |x|^2) dV \geq -(n-2) \int_{B_r} u^2 dV. \] (22)

Finally integration by parts and equation (14) give us the following identity:
\[ \int_{B_r} (| \nabla u |^2 + u b(x) \cdot \nabla u + Vu^2) (r^2 - |x|^2) dV = r \int_{\partial B_r} u^2 dS - n \int_{B_r} u^2 dV. \] (23)

Equation (23) combined with inequalities (19) and (22) shows that
\[ r \int_{\partial B_r} u^2 dS \geq \int_{B_r} | \nabla u |^2 (r^2 - |x|^2) dV - \nabla u^2 dV - (n-2) \int_{B_r} u^2 dV + n \int_{B_r} u^2 dV \]
\[ + n \int_{B_r} u^2 dV \]
\[ \geq - \int_{B_r} u^2 dV -(n-2) \int_{B_r} u^2 dV + n \int_{B_r} u^2 dV \]
\[ = \int_{B_r} u^2 dV \]
for all \( r \in (0, r_0) \) where \( r_0 \) is chosen to be the minimum of the right hand sides of inequalities (18) and (21). This completes the proof.

We give the proof of \((I)_1\) only as the proofs of \((I)_2\) and (2) are essentially the same.

**Proof of \((I)_1\):**
\[ |I_1(r)| \leq \frac{1}{r^{n-2}} \int_{B_r} |u| \Delta u |dV \]
\[ \leq \frac{1}{r^{n-2}} \int_{B_r} |u| \left( \frac{a f(r)}{r^2} |u| + \frac{b f(r)}{r} \right) \nabla u |dV \] (by (5))
\[ \leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \left( \int_{B_r} u^2 dV \right)^{1/2} \left( \int_{B_r} |\nabla u|^2 dV \right)^{1/2} \]  
(Lemma and Schwarz)

\[ \leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \frac{1}{2r} \int_{B_r} u^2 dV + \frac{r}{2} \int_{B_r} |\nabla u|^2 dV \]

\[ \leq af(r)H(r) + \frac{b}{2}f(r)D(r) \]  
(Lemma, (9) & (10))

\[ \leq C_1 f(r)(H(r) + D(r)) \]

where \( C_1 = a + b/2 \). This complete the proof of (I)₁.

A simple computation shows inequalities (I)₂, (2), (3) and (4) are satisfied with \( C_2 = a + 2b, \ C_3 = (n - 2) + (n + 2)\gamma + C_2f(r), \ C_4 = a + (3b/2)\sqrt{C_3}, \ C_5 = (3/2)C_4 \), where \( \gamma \) satisfies \( 0(\gamma) < (n + 2)\gamma \).

REFERENCES


