ERROR ESTIMATES FOR THE SEMIDISCRETE FINITE ELEMENT APPROXIMATION OF LINEAR NONLOCAL PARABOLIC EQUATIONS

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ABSTRACT

Existence and uniqueness are proved for nonlocal (in time) for solutions of linear parabolic partial differential equations. Instead of an initial condition, there is a relation connecting the initial value to values of the solution at other times. $L^2$ error estimates are obtained for the semidiscrete approximation of the problem using finite elements in the space variables.

Key words: Nonlocal parabolic equations, semidiscrete finite element approximations, error estimates.

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1. INTRODUCTION

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\Gamma$. The following nonlocal problem will be considered:

\begin{align}
  u_t + Au &= f(x,t) \text{ on } \Omega \times (0,T), \\
  u \mid_{\Gamma} &= 0, \\
  u(x,0) + g(t_1,\ldots,t_N,u) &= \psi(x),
\end{align}

where $0 < t_1 < t_2 < \ldots < t_N \leq T$, $\psi(x) \in L^2(\Omega)$, $f(x,t) \in L^\infty([0,T];L^2(\Omega))$ and $g(t_1,\ldots,t_N,\cdot)$ maps $C^0([0,T];L^2(\Omega))$ into $L^2(\Omega)$. Also assume $A$ is a strongly elliptic operator defined by

\begin{align}
  Au &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + a_0(x)u
\end{align}

with $a_{ij}(x),a_i(x) \in C^\infty(\overline{\Omega})$, with

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\end{itemize}
\[
a(u, u) \geq \sigma \| u \|^2 - \lambda_0 \| u \|^2, \quad u \in H_0^1(\Omega),
\]
where \( \sigma > 0, \lambda_0 \in R, \| u \|^2 = \| u \|^2_{L^2(\Omega)} = (u, u), \]
\[
a(u, v) = -\sum_{i, j = 1}^n \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \sum_{i = 1}^n \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} v \, dx + \int_\Omega a_0(x) uv \, dx
\]
and \( H^s(\Omega) \) and \( H^0_0(\Omega) \) are the usual Sobolev spaces with norms \( \| \cdot \|_s \). See Adams [1] or Lions [10] for definitions.

Under the above conditions, \( A \) with domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \) generates an analytic semigroup \( S(t) = e^{-At} \) such that for \( a = tr - A \circ \)
\[
\| S(t)f \| \leq Me^{-at} \| f \|,
\]
where \( M \geq 1 \) depends continuously on \( tr \) and \( A \circ \) in (1.3). See Pazy [5].

The function \( u \in C^0([0, T]; L^2(\Omega)) \) is said to be a mild solution of (1.1) if
\[
u(t) = S(t)\psi(x) - S(t)g(t_1, \ldots, t_N, u) + \int_0^t S(t - \tau)f(x, \tau) \, d\tau.
\]

We will assume for \( u, v \in C^0([0, T]; L^2(\Omega)) \) of the form \( u, v = w \), where
\[
w(t) = S(t)w(0) + \int_0^t S(t - \tau)f(x, \tau) \, d\tau,
\]
we have the Lipschitz condition
\[
\| g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, v) \| \leq \sum_{i = 1}^N m_i \| u(t_i) - v(t_i) \|.
\]

The following are some examples of \( g(t_1, \ldots, t_N, u) \): If \( h_i(x) \in C^\infty(\Omega) \), let
\[
g(t_1, \ldots, t_N, u) = \sum_{i = 1}^N h_i(x)u(t_i).
\]
The \( m_i \) in (1.8) are \( m_i = \max_{x \in \Omega} | h_i(x) | \).

Another useful example is
\[
g(t_1, \ldots, t_N, u) = \sum_{i = 1}^N \frac{1}{k_i} \int_{t_i}^{t_i + k_i} h_i(x, \tau)u(\tau) \, d\tau,
\]
where \( k_i > 0 \) and \( h_i(x, t) \in C^\infty(\Omega \times [0, T]) \). If \( u, v \) are as in (1.7) and \( t_i \leq \tau \leq t_i + k_i \), then
\[ \| u(\tau) - v(\tau) \| = \| S(\tau - t_i)(u(t_i) - v(t_i)) \| \leq M e^{-a(\tau - t_i)} \| u(t_i) - v(t_i) \|. \]

Thus the \( m_i \) in (1.8) are

\[ m_i = \frac{M}{ak_i}(1 - e^{-ak_i}) \cdot \max_{(x,t) \in \bar{\Omega} \times [t_i, t_i + k_i]} | h_i(x, t) |. \]

Nonlocal parabolic problems have been studied by several authors. See Byszewski [2-5], Chabrowski [6], Hess [7], Kerefov [8], and Vabishchevich [13].

2. EXISTENCE AND UNIQUENESS FOR NONLOCAL PROBLEMS

In this section we will prove under the conditions of section 1, (1.6) has a unique solution.

Let \( W = C([0, T]; L^2(\Omega)) \) with norm

\[ \| u \|_W = \sup_{0 \leq t \leq T} e^{at} \| u(t) \|, \]

where \( a \) satisfies (1.5). We have the following:

**Theorem 2.1:** Assume (1.5), (1.8) hold, \( \psi(x) \in L^2(\Omega) \), and \( \sum_{i=1}^{N} m_i e^{-at_i} < \frac{1}{M^2} \) for \( m_i \) in (1.8) and \( a, M \) in (1.5). Then there is a unique \( u \) in \( W \) such that \( u(t) \) satisfies (1.6).

**Proof:** Let \( \Phi: W \to W \) be defined by

\[ \Phi u(t) = S(t)\psi(x) - S(t)g(t_1, \ldots, t_N, S(t)v(0) + \int_0^t S(t - \tau)f(x, \tau)d\tau 
+ \int_0^t S(t - \tau)f(x, \tau)d\tau \]

for \( v \in W \).

We will show \( \Phi \) is a contraction mapping on \( W \). Let \( u, v \in W \). Then

\[ e^{at} \| \Phi u(t) - \Phi v(t) \| \]

\[ \leq e^{at}Me^{-at} \sum_{i=1}^{N} m_i \| S(t_i)(u(0) - v(0)) \| \]

\[ \leq \sum_{i=1}^{N} m_i M e^{-at_i} \| u(0) - v(0) \| \]

\[ \leq M^2 \left( \sum_{i=1}^{N} m_i e^{-at_i} \right) \| u - v \|_W. \]
Thus $\Phi$ is a contraction on $W$, which implies there is a unique $u \in W$ such that $u = \Phi(u)$. Since

$$u(0) = \Phi u(0) = \psi(x) - g(t_1, \ldots, t_n, S(t)u(0)) + \int_0^t S(t-\tau)f(x, \tau)d\tau$$

and

$$u(t) = S(t)u(0) + \int_0^t S(t-\tau)f(x, \tau)d\tau$$

it follows that $u(t)$ satisfies (1.6).

Since $S(t)$ has the smoothing property, $S(t)f \in D(A_{\alpha})$ for $t > 0$, $z \geq 0$ and $f \in L^2(\Omega)$, we have the following regularity property:

**Corollary 2.2:** If the conditions of Theorem 2.1 are satisfied, $\psi(x) \in D(A_{\alpha})$, $\alpha \geq 0$; $f(x, t) \in L^\infty((0, T]; D(A_{\mu}))$, $\mu = \max\{\frac{\alpha}{2} - 1 + \epsilon, 0\}$ for some $\epsilon > 0$; and $g(t_1, \ldots, t_N, \cdot)$ maps $C^0((0, T]; D(A_{\alpha}))$ into $D(A_{\alpha})$, then the solution $u(t)$ of (1.6) satisfies $u \in C^0([0, T]; D(A_{\alpha}))$.

Note: If $\sum_{i=1}^N m_i e^{-at} < \frac{1}{M^2}$ is not satisfied, there may not be a unique solution. For example, $u_t - u_{xx} + (a - \pi^2)u = 0$ on $(0, 1), u(0, t) = 0 = u(1, t)$, and $u(x, 0) - e^{-at}u(x, 1) = 0$ has solutions $u(x, t) = 0$ and $u(x, t) = e^{-at}\sin \pi x$.

### 3. THE SEMIDISCRETE APPROXIMATION

Let $\{V_h\}$ be a family of finite dimensional subspaces of $H^s(\Omega)$ such that for $f \in H^s(\Omega), 1 \leq s \leq r$,

$$\inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 \} \leq c h^s \| f \|_s,$$

where $c$ is independent of $h$.

In this section we will assume (1.3) is satisfied with $\lambda_0 = 0$. If this is not the case, let $u = e^{\lambda_0 t}W$.

For fixed $\epsilon > 0$, assume $A_h: V_h \to V_h$ satisfies

$$\langle A_h f_h, f_h \rangle \geq \sigma' \| f_h \|^2 \text{ if } f_h \in V_h,$$

where $0 < \sigma - \epsilon < \sigma' \leq \sigma$,

$$\langle A_h f_h, g_h \rangle \leq \epsilon \| f_h \|_1 \| g_h \|_1 \text{ for all } f_h, g_h \in V_h$$

and
\[ \| (P_h A^{-1} - A_h^{-1} P_h) f \| \leq ch^{\alpha+2} \| A_{\Omega}^2 f \|, \quad 0 \leq \alpha \leq r-2, \] \tag{3.4}

where \( P_h \) is the \( L^2 \) projection of \( L^2(\Omega) \) onto \( V_h \).

Conditions (3.2), (3.3) and (3.4) are satisfied with \( \sigma' = \sigma \) if the standard Galerkin method is used with \( V_h \in H^1_0(\Omega) \) and \( A_h \) is defined by

\[ (A_h f_h, g_h) = (A f_h, g_h), \quad f_h, g_h \in V_n. \]

The conditions are also satisfied if Nitsche's method is used, where \( V_h \subseteq H^1(\Omega), V_h|\Gamma \subseteq H^1(\Gamma) \), for \( 2 \leq s \leq r \),

\[ \inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 + h^{\frac{1}{2}} \| f - \chi \|_{L^2(\Gamma)} + h^{\frac{3}{2}} \| f - \chi \|_{H^1(\Gamma)} \} \leq ch^s \| f \|_s \]

and \( A_h: v_h \rightarrow v_n \) is defined by

\[ (A_h f_h, g_h) = a(f_h, g_h) - \left( \frac{\partial f_h}{\partial n}, g_h \right)_{L^2(\Gamma)} - (f_h, \frac{\partial g_h}{\partial n})_{L^2(\Gamma)} + \beta h^{-1}(f_h, g_h)_{L^2(\Gamma)} \]

for \( \beta \) large enough such that (3.2) holds. See Lasiecka [9].

We will first show the following nonlocal system on \( V_h \) has a unique solution for \( 0 \leq t \leq T \):

\[ u'_h(t) + A_h u_h = P_h f(x, t), \]

\[ u_h(0) + P_h g(t_1, \ldots, t_N, u_h) = P_h \psi. \] \tag{3.5}

Let \( S_h(t) = e^{-A_h t} \), then (3.5) is equivalent to

\[ u_h(t) = S_h(t) P_h \psi - S_h(t) P_h g(t_1, \ldots, t_N, u_h) + \int_0^t S_h(t - \tau) P_h f(x, \tau) d\tau. \] \tag{3.6}

Since \( \| e^{-A_h t} f_h \| \leq M \sigma' e^{-\sigma' t} < \frac{1}{M^2} \), where \( \lim_{\sigma' \to \sigma} M_{\sigma'} = M \), we can find \( \epsilon > 0 \) for (3.2) and \( \delta > 0 \) such that if \( m_i = m_i + \delta \) and \( \sum_{i=1}^N m_i e^{-\sigma' t_i} < \frac{1}{M^2} \), then

\[ \sum_{i=1}^N m_i e^{-\sigma' t_i} < \frac{1}{M^2}. \] \tag{3.7}

Thus by a similar proof to that of Theorem 1.1, we can prove the following:
Theorem 3.1: Assume the conditions in Theorem 1.1 are satisfied and $V_h$ and $A_h$ satisfy (3.1)–(3.4), where $\sigma'$ from (3.2) is such that (3.7) holds and

$$
\| P_h(g(t_1, \ldots, t_N, u_h) - g(t_1, \ldots, t_N, v_h)) \| \leq \sum_{i=1}^{N} m_i \| u_h(t_i) - v_h(t_i) \| \quad (3.8)
$$

for $u_h, v_h = w_n$ of the form $w_h(t) = S_h(t)w_h(0) + \int_0^t S_h(t-\tau)P_h f(x, \tau) d\tau$. Then there is a unique solution $u_h(t)$ of (3.6) such that $u_h \in C^0([0, T]; V_h)$.

Since $\| P_h(h(x)f_h) \| \leq (\sup_{x \in \Omega} | h(x) |) \| f_h \|$ for $f_h \in V_h$, if $\sigma'$ is close enough to $\sigma$, then $g$ defined in (1.9) and (1.10) satisfy (3.8).

Under the assumptions (3.1)–(3.4), we have for $a \leq s \leq r$ and $f \in D(A^2), 0 \leq \alpha \leq s$ the condition

$$
\| (S(t) - S_h(t)P_h) f \| \leq \frac{C h^s}{t^{\frac{s-\alpha}{2}}} \| A^\frac{\alpha'}{2} f \| \quad (3.9)
$$

and for $f(x, t) \in L^\infty(0, T; D(A^2)), 0 \leq \alpha' \leq r - 2$

$$
\| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h) f(x, \tau) d\tau \| \leq C h^{\alpha' + 2} \ln\left(\frac{1}{h}\right) \| f \|_{L^\infty(0, T; D(A^2))} \quad (3.10)
$$

See for example Lasiecka [9] or Thomée [12].

We can now prove similar error estimates for the semidiscrete approximation to the nonlocal problems.

Theorem 3.2: Let the assumptions of Theorems 1.1 and 3.1 be satisfied, and let the hypotheses of Corollary 2.2 be satisfied for $\alpha \leq r$, $f(x, t) \in L^\infty(0, T; D(A^2)), \theta = \max\{\mu, \alpha'\}$, $0 \leq \alpha' \leq r - 2$, and for $u, v \in C^0([t_1, T], L^2(\Omega))$, $g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, v) \| \leq k \| u - v \| L^\infty(t_1, T; L^2(\Omega))$, (3.11)

Also assume that $u(t)$ is the solution of (1.6) and $u_h(t)$ is the solution to (3.6) for $\alpha \leq s \leq r$. Then

$$
\| u(t) - u_h(t) \| \leq C h^s \left(\frac{1}{t^{\frac{s-\alpha}{2}}} + 1\right) + C h^{\alpha' + 2} \ln\left(\frac{1}{h}\right) \| f \|_{L^\infty(0, T; D(A^2))} \quad (3.12)
$$
**Proof:** We have

\[
\| u(t) - u_h(t) \| \leq \| (S(t) - S_h(t) P_h) \psi \| + \| (S(t) - S_h(t) P_h) g(t_1, ..., t_N, u) \| \\
+ \| S_h(t) P_h (g(t_1, ..., t_N, u) - g(t_1, ..., t_N, u_h)) \|
\]

\[
+ \| \int_0^t (S(t - \tau) - S_h(t - \tau) P_h) f(x, \tau) d\tau \| \tag{3.13}
\]

\[
\leq \frac{C h^\gamma}{t} \left( \| A^2 \psi \| + \| A^2 g(t_1, ..., t_N, u) \| \right) + C h^\alpha' + 2 \ln \frac{1}{h} \left\| f \right\|_{L^\infty(0, T; D(A^{1/2}) \to D(A^{1/2}))} \\
+ M e^{-\sigma' t} \| g(t_1, ..., t_N, u) - g(t_1, ..., t_N, u_h) \|.
\]

Since \( A_h \) is bounded, \( S_h(t - t) = e^{A_h t} \) exists. Let \( t \geq t_1 \), then

\[
\| g(t_1, ..., t_N, u) - g(t_1, ..., t_N, u_h) \|
\]

\[
\leq \left( \| g(t_1, ..., t_N, u) - g(t_1, ..., t_N, S_h(t - t_1) P_h S(t_1) u(0)) \| + \int_0^t \| S_h(t - \tau) P_h f(x, \tau) d\tau \| \right) \\
+ \| g(t_1, ..., t_N, S_h(t) S_h(t_1) P_h S(t_1) u(0)) \| + \int_0^t \| S_h(t - \tau) P_h f(x, \tau) d\tau \|
\]

\[
- g(t_1, ..., t_N, u_h) \| \tag{3.14}
\]

\[
\leq k \sup_{t_1 \leq t \leq T} \left( \| (S(t - t_1) - S_h(t - t_1) P_h S(t_1) u(0)) \| + \int_0^t \| (S(t - \tau) - S_h(t - \tau) P_h) f(x, \tau) d\tau \| \right) \\
+ \sum_{i=1}^N m_i \| S_h(t_i) S_h(t_1 - t_1) P_h S(t_1) u(0)) - S_h(t_i - t_1) S_h(t_1) u_h \|
\]

\[
\leq C h^\gamma \| A^2 S(t_1) u(0) \| + C h^\alpha' + 2 \ln \frac{1}{h} \left\| f \right\|_{L^\infty(0, T; D(A^{1/2}))} \\
+ \sum_{i=1}^N m_i \| S(t_1) u(0) - S_h(t_1) u_h(0) \|
\]

\[
\leq C h^\gamma \| A^2 S(t_1) u(0) \| + C h^\alpha' + 2 \ln \frac{1}{h} \left\| f \right\|_{L^\infty(0, T; D(A^{1/2}))} \\
+ \sum_{i=1}^N m_i \| S(t_1) u(0) - S_h(t_1) u_h(0) \|
\]

\[
\leq C h^\gamma \| A^2 S(t_1) u(0) \| + C h^\alpha' + 2 \ln \frac{1}{h} \left\| f \right\|_{L^\infty(0, T; D(A^{1/2}))} \\
+ \sum_{i=1}^N m_i \| S(t_1) u(0) - S_h(t_1) u_h(0) \|
\]

\[
+ \sum_{i=1}^N m_i \| S(t_1) u(0) - S_h(t_1) u_h(0) \|
\]
\[ \leq Ch^g \| A_2^T S(t_1)u(0) \| + Ch^{\alpha'} + 2\ln \frac{1}{h} \| f \|_{L^\infty(0,T;D(A^2))} \]
\[ + \sum_{i=1}^N m_i \sigma_i e^{-\sigma'(t_i - t_1)} \| u(t_1) - u_h(t_1) \|. \]

Let \( t = t_1 \) in (3.13), then

\[ \| u(t_1) - u_h(t_1) \| \leq C\left( \frac{h^g}{t_1^{\frac{a}{2}}} + 1 \right) + Ch^{\alpha'} + 2\ln \frac{1}{h} \| f \|_{L^\infty(0,T;D(A^2))} \]
\[ + M^2 \sum_{i=1}^N m_i \sigma_i e^{-\sigma'(t_i - t_1)} \| u(t_1) - u_h(t_1) \|. \]

(3.15)

Since \( M^2 \sum_{i=1}^N m_i \sigma_i e^{-\sigma'(t_i - t_1)} < 1 \), (3.12) holds for \( t = t_1 \). Therefore the theorem follows from (3.13) and (3.15).

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