AN ABSTRACT INVERSE PROBLEM

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ABSTRACT

In this paper we consider an inverse problem that corresponds to an abstract integrodifferential equation. First, we prove a local existence and uniqueness theorem. We also show that every continuous solution can be locally extended in a unique way. Finally, we give sufficient conditions for the existence and a stability of the global solution.

Key words: inverse problem, abstract integrodifferential equation, existence, uniqueness, stability.

AMS (MOS) subject classification: 351%.

1. INTRODUCTION

Let \( X, Y \) be two Banach spaces, and let \( A: D(A) \subseteq X \to X \) be a linear operator. Let \( T > 0, F_1, F_2: [0, T] \times X \times Y \to X, L: X \to Y, v: [0, T] \to Y \), and \( z \in X \) be given data.

We consider the following problem: find \( (u, p): [0, T] \to X \times Y \) such that

\[
\begin{align*}
  u'(t) &= Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s)) ds, \quad 0 \leq t \leq T, \\
  u(0) &= z, \\
  Lu(t) &= v(t), \quad 0 \leq t \leq T.
\end{align*}
\]

Such a problem has been considered previously by Prilepko, Orlovskii in [6,7], Lorenzi, Sinestrari in [4], and the author in [1].

The local existence and uniqueness result is obtained by Prilepko, Orlovskii for the case \( F_2 = 0 \), and by Lorenzi, Sinestrari for the case \( Y \) is a subspace of \( L(X) \), \( F_1(t, u, p) = pBz \), and \( F_2(t, u, p) = pBu \), where \( B \) is some given linear operator in \( X \). The stability problem has been studied by Lorenzi and Sinestrari in [5].

In [1] the author treats the case of \( Y = C[0, T]^n (n \geq 1), F_1(t, u, (p_1, \ldots, p_n)) = \sum_1^n p_i y_i, \; y_i \in X(1 \leq i \leq n) \) and \( F_2 = 0 \). Then a global existence and uniqueness theorem is obtained.

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The present work is concerned with a generalization of those results.

Throughout this paper we assume:

\( (H1) \)  
A is a closed linear operator with a dense domain generating a strongly continuous semigroup \( e^{At} \). Without loss of generality, we suppose that \( e^{At} \) is equibounded:

\[ \| e^{At} \| \leq M, t \geq 0 \] for some \( M \geq 1 \).

\( (H2) \)  
\( x \in D(A) \),

\( (H3) \)  
\( L \in L(X,Y) \),

\( (H4) \)  
\( v \in C^1([0,T]: Y) \), and \( v(0) = Lx \).

\( (H5,1) \)  
\( F_1 \) and \( AF_1 \) are continuous in \([0,T] \times D(A) \times Y \).

For each \( r > 0 \), there exist positive continuous real valued functions \( g_{1,i}(r, \cdot) \), \( i = 0,1 \) such that

\( (H5,2) \)  
\[ \| F_1(t, u_1, p_1) \|_{D(A)} \leq g_{1,0}(r, t), \]

\( (H5,3) \)  
\[ \| F_1(t, u_1, p_1) - F_1(t, u_2, p_2) \|_{D(A)} \leq g_{1,1}(r, t)(\| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y), \]

for each \((u_i, p_i) \in \{(u, p) \in D(A) \times Y, \| u \|_{D(A)} + \| p \|_Y \leq r \}, i = 1, 2, \) and \( t \in [0, T] \).

\( (H6,1) \)  
\[ \int_0^t F_2 \text{ and } A \int_0^t F_2 \text{ are continuous in } [0, T] \times D(A) \times Y. \]

For each \( r > 0 \), there exist positive continuous real valued functions \( g_{2,i}(r, \cdot) \), \( i = 0,1 \), such that

\( (H6,2) \)  
\[ \| \int_0^t F_2(s, u_1(s), p_1(t-s))ds \|_{D(A)} \leq \int_0^t g_{2,0}(r, s)ds, \]

\( (H6,3) \)  
\[ \| \int_0^t \left( F_2(s, u_1(s), p_1(t-s)) - F_2(s, u_2(s), p_2(t-s)) \right)ds \|_{D(A)} \]
\[ \leq \int_0^t g_{2,1}(r, s)(\| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y)ds, \]

for each \((u_i, p_i) \in \{(u, p) \in C([0,T]: D(A) \times Y), \sup_{0 \leq s \leq t} (\| u(s) \|_{D(A)} + \| p(s) \|_Y) \leq r \}, i = 1, 2, \) and \( t \in [0, T] \).

There exist continuous function \( H_1: [0,T] \times Y \times Y \rightarrow Y \) with the following properties. For each \( r > 0 \) there exist positive continuous real valued functions \( C(r, \cdot) \) such that

\( (H7,1) \)  
\[ \| H_1(t, u_1, p_1) - H_1(t, u_2, p_2) \|_Y \leq C(r, t)(\| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y), \]

for each \((u_i, p_i) \in \{(u, p) \in Y \times Y, \| u \|_Y + \| p \|_Y \leq r \}, i = 1, 2, \) and \( t \in [0, T] \).

\( K: p \rightarrow H_1(t, v(t), p) \) has an inverse \( \Phi(t, \cdot) \) continuous map \( t \mapsto \Phi(t, w) \), and there exist positive continuous real valued function \( k \), such that
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\[ (H7, 2) \quad \| \Phi(t, w_1) - \Phi(t, w_2) \|_Y, t \in [0, T], w_i \in Y, i = 1, 2. \]

\[ (H7, 3) \quad LF_1(t, u, p) = H_1(t, Lu, p), (u, p) \in D(A) \times Y, \text{ and } t \in [0, T]. \]

2. **Existence of the Local Solution**

In this section we prove that the local solution of our inverse problem is obtained by a fixed point theorem. Let

\[ a(t) = M \| x \|_{D(A)} + \| \Phi(t, 0) \|_Y + k(t) \| v'(t) - Le^{At}Az \|_Y, t \in [0, T], r_0 = 2 \sup_{0 \leq t \leq T} a(t), \]

\[ g_i(r_0, t, s) = M(1 + k(t) \| L \|)(g_{1,i}(r_0, s) + (t - s)g_{2,i}(r_0, s)) + k(t) \| L \| g_{2,i}(r_0, s), \quad 0 \leq s \leq t \leq T, \]

\[ i = 0, 1, \text{ and let } T_0 \in [0, T] \text{ be such that} \]

\[ T_0 \sup_{0 \leq s \leq t \leq T} g_0(r_0, t, s) \leq \frac{r_0}{2}, \text{ and } T_0 \sup_{0 \leq s \leq t \leq T} g_1(r_0, t, s) = \gamma < 1. \]

Let \( Z(T_0) = C([0, T_0]; D(A) \times Y) \) equipped with the norm

\[ \| (u, p) \|_{Z(T_0)} = \sup_{0 \leq t \leq T_0} (\| u(t) \|_{D(A)} + \| p(t) \|_Y). \]

Then, we define the mapping

\[ \Psi: Z(T_0) \to Z(T_0); (u, p) \mapsto (U, P), \]

where

\[ U(t) = e^{At}x + \int_0^t e^{A(t-s)}F_1(s, u(s), p(s))ds \]

\[ + \int_0^t e^{A(t-s)} \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma))d\sigma ds, \]

\[ P(t) = \Psi(t, v'(t) - Le^{At}Az - \int_0^t LF_2(s, u(s), p(t-s))ds \]

\[ - \int_0^t Le^{A(t-s)}AF_1(s, u(s), p(t-s))ds \]

\[ - \int_0^t Le^{A(t-s)}A \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma))d\sigma ds, 0 \leq t \leq T_0. \]

**Proposition 1.** There exists a unique \((u_0, p_0)\) in \(B(r_0, T_0)\) satisfying \((u_0, p_0) = \Psi(u_0, p_0)\),

where \(B(r_0, T_0)\) denotes the closed ball of \(Z(T_0)\) with the center 0 and radius \(r_0\).
Proof. We claim that $\Psi$ is a strict contraction from $B(r_0, T_0)$ into itself. Hence, according to the fixed point theorem, there is a unique $(u_0, p_0)$ in $B(r_0, T_0)$ such that $(u_0, p_0) = \Psi(u_0, p_0)$.

Let $(u_i, p_i)$ in $B(r_0, T_0), (U_i, P_i) = \Psi(u_i, p_i), i = 1, 2,$ and $t$ in $[0, T_0]$.

We have then

$$\| U_1(t) \|_{D(A)} \leq M \| x \|_{D(A)} + M \int_0^t \| F_1(s, u(s), p(s)) \|_{D(A)} ds$$

$$+ M \int_0^t \| \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma \|_{D(A)} ds.$$

Using (H5) and (H6) we obtain

$$\| U_1(t) \|_{D(A)} \leq M \| x \|_{D(A)} + M \int_0^t g_{1,0}(r_0, s) ds + M \int_0^t g_{2,0}(r_0, \sigma) d\sigma ds$$

$$\leq M \| x \|_{D(A)} + M \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds.$$

From (H7,2) we deduce

$$\| P_1(t) \|_{Y} \leq \| \Phi(t, 0) \|_{Y} + k(t) \| v'(t) - L e^{At}Az \|$$

$$- \int_0^t L F_2(s, u(s), p(t-s)) ds - \int_0^t L e^{A(t-s)} A F_1(s, u(s), p(t-s)) ds$$

$$- \int_0^t L e^{A(t-s)} A \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma ds \|_{Y}.$$

Hence

$$\| P_1(t) \|_{Y} \leq \| \Phi(t, 0) \|_{Y} + k(t) \| v'(t) - L e^{At}Az \| + \| L \| k(t) \int_0^t g_{2,0}(r_0, s) ds$$

$$+ M \| L \| k(t) \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds.$$

Thus

$$\| U_1(t) \|_{D(A)} + \| P_1(t) \|_{Y} \leq \| x \|_{D(A)} + \| \Phi(t, 0) \|_{Y} + k(t) \| v'(t) - L e^{At}Az \|$$

$$+ \| L \| k(t) \int_0^t g_{2,0}(r_0, s) ds$$

$$+ M(1 + \| L \| k(t)) \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds$$

$$\leq a(t) + \int_0^t g_1(r_0, t, s) ds.$$
This implies that
\[ \| (U_1, P_1) \|_{Z(T_0)} \leq \tau_0. \]

On the other hand, in the same way as above, it is easily seen that
\[ \| U_1(t) - U_2(t) \|_{D(A)} + \| P_1(t) - P_2(t) \|_Y \]
\[ \leq \int_0^t g_2(\tau_0, t, s) \left( \| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y \right) ds \]
\[ \leq \gamma \sup_{0 \leq s \leq t} \left( \| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y \right) ds. \]

It follows that
\[ \| (U_1, P_1) - (U_2, P_2) \|_{Z(T_0)} \leq \gamma \| (u_1, p_1) - (u_2, p_2) \|_{Z(T_0)}. \]

Our claim is proven.

**Proposition 2.** \((u, p)\) is a solution of the inverse problem (1) – (3) in \([0, T]\) iff \((u, p) = \Psi(u, p)\).

**Proof.** It is well known that the solution of Cauchy problem (1) and (2) is given by \(u(t) = U(t)\). Therefore, it suffices to show
\[ Lu(t) = v(t) \text{ iff } p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t)) \]
for each \(t\) in \([0, T]\).

First, we differentiate \(Lu(t) = v(t)\) to obtain
\[ Lu'(t) = L\{Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s)) ds\} = v'(t). \]

Hence
\[ H_1(t, v(t), p(t)) = LF_1(t, u(t), p(t)) \]
\[ = v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t). \]

Using \((H7, 2)\) we get
\[ p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s)) ds - LAu(t)). \]

Conversely, this last equality implies that
\[
H_1(t, v(t), p(t)) = v'(t) - L\left( \int_0^t F_2(s, u(s), p(t-s)) ds - Au(t) \right)
- v'(t) - L\{v'(t) - F_1(t, u(t), p(t))\}
= v'(t) - Lu'(t) + H_1(t, Lu(t), p(t)).
\]

Thus
\[
\frac{d}{dt}(v(t)) = H_1(t, v(t), p(t)) - H_1(t, Lu(t), p(t)).
\]

Integrating and using the fact that \(v(0) = Lu(0) = Lx\), we obtain
\[
v(t) - Lu(t) = \int_0^t (H_1(s, v(s), p(s)) - H_1(s, Lu(s), p(s))) ds.
\]

But, \((H7, 1)\) leads to
\[
\|v(t) - Lu(t)\|_Y = \int_0^t C(R, s) \|v(s) - Lu(s)\|_Y ds,
\]

where
\[
R = \max( \sup_{0 \leq t \leq T} (\|Lu(t)\|_Y + \|p(t)\|_Y), \sup_{0 \leq t \leq T} (\|v(t)\|_Y + \|p(t)\|_Y)).
\]

Hence, by using Gronwall’s inequality, it follows that
\[
v(t) - Lu(t) = 0, \ 0 \leq t \leq T.
\]

Now, we combine propositions 1 and 2 to deduce the following local existence and uniqueness theorem for the inverse problem \((1)-(3)\).

**Theorem 1.** Under the assumptions \((H1)-(H7)\), there exist \(T_0\) in \([0, T]\) and \((u_0, p_0)\) in \(C([0, T_0]; D(A) \times Y)\) which is the unique solution of the inverse problem \((1)-(3)\) in \([0, T_0]\).

**Remark.** Theorem 1 is still valued if we add to the right side of equality \((1)\) a function \(f: [0, T] \rightarrow X\) such that \(f\) and \(Af\) are continuous.

### 3. GLOBAL SOLUTION

We begin this section by showing that any solution \((u_0, p_0)\) in \(C([0, T_0]; D(A) \times Y)\) of the inverse problem \((1)-(3)\) in \([0, T_0]\) can be uniquely extended to a solution in \([0, T_0 + T_1]\) for some \(T_1 > 0\), whenever \(0 < T_0 < T\).

If \(T\) is in \([0, \min(T_0, T - T_0)]\), we consider the following inverse problem:
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where

\[ u'(t) = Au(t) + K_1(t, u(t), p(t)) + \int_0^t K_2(s, u(s), p(t-s))ds + f(t), \quad 0 \leq t \leq T \]

\[ u(0) = \pi_1 = u_0(T_0) \]

\[ Lu(t) = w(t), \; 0 \leq t \leq T \]

\[ K_1(t, u(t), p(t)) = F_1(t + T_0, u(t), p(t)), \; 0 \leq t \leq T, \]

\[ K_2(s, u(s), p(t-s)) = F_2(s, u_0(s), p(t-s)) + F_2(s + T_0, u(s), p_0(t-s)), \; 0 \leq t \leq T, \]

\[ f(t) = \int_t^T F_2(s, u_0(s), p_0(t-s))ds, \; 0 \leq t \leq T, \text{ and} \]

\[ w(t) = v(t + T_0), 0 \leq t \leq T. \]

**Proposition 3.** If \((u_0, p_0)\) in \(C([0,T_0];D(A) \times Y)\) denotes any solution of the inverse problem (1)-(3) in \([0,T_0]\), then there exist \(T_1 \in [0, \min(T_0, T - T_0)]\) and \((u, p)\) in \(C([0,T_0 + T_1];D(A) \times Y)\) such that \((u, p) = (u_0, p_0)\) in \([0,T_0]\), and \((u, p)\) satisfies (1)-(3) in \([0,T_0 + T_1]\).

**Proof.** It is not difficult to see that \(K_1, K_2, w\) have the same properties as \(F_1, F_2, v, p\), and that \(f\) and \(Af\) are continuous. It follows from Theorem 1 that there exist \(T_1 \in [0, T]\) and \((u_1, p_1) \in C([0,T_1];D(A) \times Y)\), which is the unique solution of the inverse problem (4)-(6) given by

\[ u_1(t) = e^{At}x + \int_0^t e^{A(t-s)}K_1(s, u(s), p(s))ds + \int_0^t e^{A(t-s)}f(s)ds \]

\[ + \int_0^t e^{A(t-s)} \int_0^s K_2(\sigma, u(\sigma), p(s-\sigma))d\sigma d\sigma, \; 0 \leq t \leq T_1, \]

\[ p_1(t) = \Psi(t + T_0, w'(t) - LAu_1(t) - \int_0^t LK_2(s, u_1(s), p_1(t-s))ds - Lf(t)), \; 0 \leq t \leq T_1. \]

We have

\[ p_1(0) = \Psi(T_0, w'(0) - LAu_1(0) - Lf(0)) \]

\[ = \Psi(T_0, v'(0) - LAu(T_0) - \int_0^T LK_2(s, u_0(s), p_0(T_0 - s))ds \]

\[ = p(T_0). \]
One can easily check that
\[
(u(t), p(t)) = \begin{cases} 
(u_0(t), p_0(t)), & 0 \leq t \leq T_0, \\
(u_1(t), p_1(t)), & T_0 < t \leq T_1, 
\end{cases}
\]
belongs to \(C([0, T_0 + T_1]; D(A) \times Y)\). It remains to show that \((u, p)\) is a solution of the inverse problem (1)–(3) in \([0, T_0 + T_1]\). Since \(u_1\) satisfies (4), we can deduce that
\[
u'(t + T_1) = u_1'(t)
\]
\[
= Au_1(t) + F_1(t + T_0, u_1(t), p_1(t)) + \int_0^t F_2(s, u_0(s), p_1(t - s))ds
\]
\[
+ \int_0^{T_0} F_2(s + T_0, u_1(s), p_0(t - s))ds + \int_0^t F_2(s, u_0(s), p_0(t + T_0 - s))ds
\]
\[
= Au(t + T_0) + F_1(t + T_0, u(t + T_0), p(t + T_0)) + \int_0^t F_2(s, u(s), p(t + T_0 - s))ds
\]
\[
+ \int_0^{t + T_0} F_2(s, u(s), p(t + T_0 - s))ds + \int_0^t F_2(s, u(s), p(t + T_0 - s))ds
\]
\[
= Au(t + T_0) + F_1(t + T_0, u(t + T_0), p(t + T_0))
\]
\[
+ \int_0^{t + T_0} F_2(s, u(s), p(t + T_0 - s))ds, \quad 0 \leq t \leq T_1.
\]

On the other hand
\[
Lu(t + T_0) = L u_1(t) = w(t) = v(t + T_0), \quad 0 \leq t \leq T_1.
\]

Therefore we may conclude that \((u, p)\) is a solution of the inverse problem (1)–(3) in \([0, T_0 + T_1]\).

**Proposition 4.** Let \((u, p) \in C([0, T_{\max}; D(A) \times Y)\) be the maximal solution of the inverse problem (1)–(3), where \(0 < T_{\max} \leq T\). If

\[
(7) \quad \max_{0 < t < T_{\max}} \left( \sup_{0 \leq s \leq t} \left( \| u(s) \|_{D(A)} + \| p(s) \|_{Y} \right) \right) < +\infty,
\]

then \(T_{\max} = T\).

**Proof.** Clearly, from Proposition 2 \((u, p)\) can be continuously extended to a solution in
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[0, T_{\text{max}}]. If T_{\text{max}} < T, then, following the previous proposition, the solution in [0, T_{\text{max}}] can be extended to a solution in [0, T_{\text{max}} + \epsilon], for some \epsilon > 0. This contradicts the maximality of T_{\text{max}}.

Now, we will give a sufficient conditions to realize (7). For this purpose, we recall the following comparison theorem.

**Theorem 2** [2]. Let I be a real interval, and let \( G: I \times I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be continuous such that \( G(t,s,r) \) is monotone nondecreasing in \( r \) for each \((t,s)\) in \( I \times I \). Let \( b \) in \( C(I) \), and let \( f \) in \( C(I) \) denote the maximal solution of the integral equation

\[
f(t) = b(t) + \int_{t_0}^{t} G(t,s,f(s))ds, \quad t \geq t_0.
\]

If \( g \in C(I) \) is such that

\[
g(t) \leq b(t) + \int_{t_0}^{t} G(t,s,g(s))ds, \quad t \geq t_0,
\]

then \( g(t) \leq f(t), t \geq t_0 \).

Here, by a maximal solution we mean that any other solution \( h \in C(I) \) must satisfy \( h(t) \leq f(t), t \geq t_0 \).

Before stating a global existence and uniqueness result for our inverse problem, we need to modify some assumptions on \( F_1 \), and \( F_2 \).

Instead of \((H5,2)\) and \((H6,2)\) we suppose that there exist \( G_i(t,r):[0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) continuous and monotone nondecreasing in \( r \) for each \( t \) in \([0, T]\), \( i = 1, 2 \), such that

\[
(H5,2') \quad \| F_1(t,u,p) \|_{D(A)} \leq G_1(t,\| u \|_{D(A)} + \| p \|_{Y}),
\]

\[
(H6,2') \quad \| \int_{0}^{t} F_2(s,u(s),p(t-s))ds \|_{D(A)} \leq \int_{0}^{t} G_2(s,\| u(s) \|_{D(A)} + \| p(s) \|_{Y})ds.
\]

Set

\[
G(t,s,r) = M(1 + k(t)\| L \|)(G_1(s,r) + (t-s)G_2(s,r)) + k(t)\| L \| G_2(s,r), 0 \leq s \leq t \leq T; i = 0, 1.
\]

Clearly, \( G(t,s,r) \) is monotone nondecreasing in \( r \), \( 0 \leq s \leq t \leq T \).

**Theorem 3.** Assume that \((H1) - (H7)\) are satisfied, where \((H5,2)\) and \((H6,2)\) are changed by \((H5,2')\) and \((H6,2')\). If the nonlinear Volterra integral equation:
(8) \[ r(t) = a(t) + \int_0^t G(t, s, r(s))ds, \quad 0 \leq t \leq T, \]

has a continuous maximal solution in \([0, T]\), then the inverse problem (1) – (3) has a unique solution in \([0, T]\).

**Proof.** Let \( r \) denote the continuous maximal solution of the integral equation (8). Proceeding in the manner of the proof of Proposition 1, we obtain

\[
\| u(t) \|_{D(A)} + \| p(t) \|_Y \leq a(t) + \int_0^t G(t, s, \| u(s) \|_{D(A)} + \| p(s) \|_Y)ds, \quad 0 \leq t \leq T.
\]

Thus, the condition (7) is satisfied.

The uniqueness of the global solution is just a consequence of the fact that the unique local solution allows a unique extension.

**4. STABILITY RESULT**

First of all, we give the exact assumptions under which the stability result will hold.

We assume that \((H1) – (H5, 1), (H6, 1), (H7)\) are satisfied, and there exist \( G_i(t, r): [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+ \) continuous and monotone nondecreasing in \( r \) for each \( t \) in \([0, T]\), \( i = 1, 2 \), such that

\[(H8, 1) \quad \| F_1(t, u_1, p_1) - F_1(t, u_2, p_2) \|_{D(A)} \leq G_1(t, \| u_1 - u_2 \|_{D(A)} + \| p_1 - p_2 \|_Y), \quad \text{for each} \quad (u_i, p_i) \in D(A) \times Y, i = 1, 2, \quad \text{and} \quad 0 \leq t \leq T.\]

\[(H8, 2) \quad \| \int_0^t (F_2(s, u_1(s), p_1(t-s)) - F_2(s, u_2(s), p_2(t-s)))ds \|_{D(A)} \leq \int_0^t G_2(s, \| u_1(s) - u_2(s) \|_{D(A)} + \| p_1(s) - p_2(s) \|_Y)ds\]

for each \((u_i, p_i) \) in \( C([0, T]; D(A) \times Y), i = 1, 2, \) and \( 0 \leq t \leq T.\)

\((v(t), H_1(t, v(t), p)) - \Phi(t, K(p))\) has the following property:

there exist continuous \( g: [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+ \), such that

\[\| \Phi_1(t, w_1) - \Phi_2(t, w_2) \|_Y \leq g(t)(\| v_1(t) - v_2(t) \|_Y + \| w_1 - w_2 \|_Y),\]

for each \( v_i \) in \( C([0, T]; Y), w_i \in Y, i = 1, 2, \) and \( 0 \leq t \leq T.\)
Here, \( \Phi_i(t, \cdot) \) denotes the inverse of the mapping \( K_i: \Phi \rightarrow H_i(t, v(t), \Phi) \), \( i = 1, 2 \). We set

\[
G(t, s, r) = M(1 + g(t) ||L||)(G_1(s, r) + (t - s)G_2(s, r)) + g(t) ||L||G_2(s, r),
\]

\[0 \leq s \leq t \leq T, \quad i = 0, 1.\]

**Theorem 4.** Suppose that the assumptions listed below are satisfied for \( z = z_i, \quad v = v_i, \quad i = 1, 2 \). Let \( (u_i, p_i) \) in \( C([0, T]; D(A) \times Y) \) denote any solution of the inverse problem (1)-(3) corresponding to \( x = z_i, \quad v = v_i, \quad i = 1, 2 \), and let

\[
r_0(t) = M(1 + g(t) ||L||) ||z_1 - z_2||_{D(A)} + n(t)(||v_1(t) - v_2(t)|| Y + (||v'_1(t) - v'_2(t)|| Y)).
\]

If the maximal continuous solution, given its existence, of the Volterra integral equation

\[
m(t) = r_0(t) + \int_0^t G(t, s, m(s))ds, \quad 0 \leq t \leq T,
\]

satisfies the condition that there exists a constant \( C > 0 \), not depending on \( m \), such that

\[
m(t) \leq Cr_0(t), \quad 0 \leq t \leq T,
\]

then

\[
\|u_1(t) - u_2(t)\|_{D(A)} + ||p_1(t) - p_2(t)|| Y \leq Cr_0(t), \quad 0 \leq t \leq T.
\]

**Proof.** Let \( m \) denote the maximal solution of the integral equation (9), and let

\[
r(t) = \|u_1(t) - u_2(t)\|_{D(A)} + ||p_1(t) - p_2(t)|| Y, \quad 0 \leq t \leq T.
\]

It is easy to see that

\[
r(t) \leq r_0(t) + \int_0^t G(t, s, m(s))ds, \quad 0 \leq t \leq T.
\]

Using the comparison Theorem 2, we deduce that \( r(t) \leq m(t) \). Hence, (11) follows from (10).

**Remark.** We have \( G(t, s, r) \leq G(T, s, r) \). Then if \( G(T, s, r) \) takes the form \( G(T, s, r) = G(s)r \), the conclusion of Theorem 4.1 follows from Gronwall's inequality.
REFERENCES


