STABILITY OF VOLterra SYSTEM WITH IMPULSIVE Effect¹

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ABSTRACT

Sufficient conditions for uniform stability and uniform asymptotic stability of impulsive integrodifferential equations are investigated by constructing a suitable piecewise continuous Lyapunov-like functionals without the decrent property. A result which establishes no pulse phenomena in the given system is also discussed.

Key words: Uniform stability, asymptotic stability beating, Lyapunov functional, fundamental matrix, integral curves, surfaces.

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1. **INTRODUCTION:** The stability analysis of ordinary differential equations with impulsive effect has been the subject of many investigations [1, 2, 4] in recent years and various interesting results are reported. However, much has not been developed in this direction of integro-differential equations with impulsive effect except for a few [3, 5] in which the impulsive integral inequalities are used. The purpose of this paper is to investigate sufficient conditions for uniform stability and uniform asymptotic stability of Linear integro-differential equations by employing the piecewise continuous Liapunov functional without the decrescent property. It is also proved that every solution of the integro-differential system meets any given surface exactly once and thus there exists no pulse phenomena in the system.

Let the hyper surfaces $\sigma_k$ be defined by the equations

$$\sigma_k = t = \tau_k(x), \ 0 < \tau_1(x) < \ldots < \tau_k(x) < \ldots$$

where $\tau_k(x) \rightarrow \infty$ as $k \rightarrow \infty$.

$Pc^+$ denote the class of piecewise continuous functions from

$$\mathbb{R}_+^2 \rightarrow \mathbb{R}^{n^2}$$

with discontinuities of the first kind at $t \neq \tau_k(x), k = 1, 2, \ldots$ and left continuous at $t = \tau_k$.

Let $\tau_0(x) = 0$ for $x \in \mathbb{R}_+$ and

$$G_k = \{(t,x) \in I \times \mathbb{R}^n: \tau_{k-1}(x) < t < \tau_k(x)\}, \ k = 1, 2, \ldots$$

The function $V: I \times \mathbb{R}^+ \rightarrow \mathbb{R}$ belongs to class $V_0$ if:

(i) The function $V$ is continuous on each of the sets $G_k$ and $V(t,0) = 0$
(ii) For each \( k = 1, 2, \ldots \) and \((t_0, x_0) \in G_k\) there exists finite limits

\[
V(t_0 - 0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x); \quad V(t_0, x_0) = \lim_{(t, x) \to (t_0, x_0)} V(t, x)
\]

and \( V(t_0 - 0, x_0) = V(t_0, x_0) \) is satisfied.

Also if \((t_0, x_0) \in G_k\) then \( V(t_0 + 0, x_0) = V(t_0, x_0) \)

Let \( V \in V_0 \) For \((t, x) \in \mathbb{U}G_k\), \( D^+ V \) is defined as

\[
D^+ V(t, x) = \lim_{h^{-} \to 0^+} \sup_{h} \frac{1}{h} [V(t + h, x(t + h)) - V(t, x(t))]
\]

2. Consider the impulsive integro-differential system

\[
x' = (A(t)x + \int_{t_0}^{t} k(t, s)x(s) \, ds) \quad , t \neq \tau_k(x), k = 1, 2, \ldots
\]

\[
\Delta x|_{t=\tau_k(x)} = I_k(x) \quad , x(t_0) = x_0
\]

(2.1)

where \( A \in PC^+ [R_+, R^{n_2}], K \in PC^+[R^2_+, R^{n_2}], \) and \( I_k(0) = 0, t \geq t_0, k = 1, 2, \ldots \)

Let us consider:

\[
x' = A(t)x \quad , t \neq \tau_k(x)
\]

\[
\Delta x|_{t=\tau_k(x)} = B_k(x)
\]

(2.2)

where \( \det (I + B_k) \neq 0. \)

Not let \( \phi_k(t, s) \) be the fundamental matrix of the linear system

\[
x' = A(t)x, \quad (\tau_{k-1} < t < \tau_k)
\]

(2.3)
Then the solution of the linear system (2.2) can be written in the form

\[ x(t, t_0, x_0) = \psi(t, t_0 + 0)x_0, \]

where

\[
\psi(t, s) = \begin{cases} 
\phi_k(t, s) & \text{for } t_{k-1} < s < t_k \\
\phi_{k+1}(t, t_k)(I+B_k)\phi_k(t_k, s) & \text{for } t_{k-1} < s < t_k < t_{k+1} \\
\phi_k(t, t_k)(I+B_k)^{-1}\phi_{k+1}(t_k, s) & \text{for } t_{k-1} < s < t_k < t_{k+1}
\end{cases}
\]

The following Lemma gives sufficient conditions for the absence of beating.

**Lemma 2.1:** Let the following conditions be satisfied for \(|x| < \rho\)

(i) \(|\phi_k(t, s)| \leq \alpha e^\lambda (t-s)\) for \(0 \leq s < t < \infty\) for all \(k\).

(ii) \(|A(t)| \leq \beta\) for \(t \geq 0\).

(iii) \(|(I + B_k)| \leq \gamma\) where \(I\) is the identity matrix.

(iv) \(|K(t, s)| \leq M e^{\sigma (t-s)}\) where \(M > 0, \sigma > 0\) for \(0 \leq s < t < \infty\)

(v) There exists a number \(\bar{h} > 0\) such that

\[
\sup_{0 \leq s \leq 1} \frac{\partial \tau_k}{\partial x}(x+sI_k(x)) \leq 0, k=1,2,\ldots \\
|x| \leq \bar{h}
\]

and

\[
\sup_{|x| < \bar{h}} \left| \frac{\partial \tau_k(x)}{\partial x} \right| \leq N, \ k=1,2,\ldots
\]

(vii) \((\beta + \frac{M}{\sigma})\rho N < 1\)
Then there exists a number \( \rho \leq \bar{\alpha} \) such that if \( x(t) \) is a solution of (2.1), which lies in the ball \( \{ x \in \mathbb{R}^n : |x| \leq \rho \} \) for \( 0 \leq t \leq T, T > 0 \), then the integral curve

\[ \{(t, x(t)) : t \in [0, T]\} \]

meets the hyper surface \( t = \tau_k(x) \) exactly once.

**Proof:** Let \( F(t, s) = A(t) x + \int_{t_0}^{t} K(t, s) x(s) \, ds \)

If \( |x| \leq \rho \) then from (2.1) and (i), (ii), (iii), and (iv) we get

\[
|F(t, s)| \leq |A(t) x| + \int_{t_0}^{t} |K(t, s)| |x(s)| \, ds
\]

\[
\leq \beta |x| + M \int_{t_0}^{t} \theta \sigma(t-s) \sup_{0 \leq s \leq T} |x(s)| \, ds
\]

\[
\leq \beta |x| + M \rho \int_{t_0}^{t} \theta \sigma(t-s) \, ds
\]

\[
\leq (\beta + \frac{M}{\sigma}) \rho
\]

Now assume that some solution \( x(t) \) of (2.1) under the above assumptions meets some surface \( t = \tau_k(x) \) more than once.

Let \( t = t_j \) be the point at which the solution first meets the surface \( t = \tau_k(x) \) for some \( j \) and again another closest hit at \( t = t^* \) such that \( t^* - t_j > 0 \). Then we have

\[ t_j = \tau_k(x(t_j)) \quad \text{and} \quad t^* = \tau_k(x(t^*)) \quad \text{where} \quad t_0 < t_j < t^* \]

Then the solution satisfies the integral equation

\[
x(t) = x_j + I_k(x_j) + \int_{t_j}^{t} F(s, x(s)) \, ds
\]
Let \( h = \int_{t_j}^{t^*} F(s, x(s)) \, ds \).

Define the function \( x(s) = \tau_k(x_j + I_k(x_j) + sh) + \tau_k(x_j + sI_k(x_j)) \)
for \( s \in [0,1] \). Then by mean value theorem
\[
x(1) - x(0) = \int_0^1 x'(s) \, ds
\]

\[
t^* - t_j = \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + h) - \tau_k(x_j)
= \int_0^1 \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh) \, ds \cdot (I_k(x_j) + h)
= \int_0^1 \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh) \cdot h \, ds + \int_0^1 \frac{\partial \tau_k}{\partial x}(x_j + sI_k(x_j)) \cdot I_k(x_j) \, ds
\]

(2.4)

Since we have \( \frac{\partial \tau_k(x)}{\partial x} \leq N \) and \( |F(s, x(s))| \leq (\beta + \frac{M}{\sigma}) \rho \)
By Cauchy-Schwartz inequality the first integral on the right hand side of (2.4) satisfies
\[
\int_0^1 \frac{\partial \tau_k}{\partial x}(x_j + I_k(x_j) + sh) \cdot h \, ds \leq N(\beta + \frac{M}{\sigma}) \rho (t^* - t_j)
\]
hence we have
\[
[1 - N(\beta + \frac{M}{\sigma}) \rho](t^* - t_j) \leq \int_0^1 \frac{\partial \tau_k}{\partial x}(x_j + sI_k(x_j)) \cdot I_k(x_j) \, ds
\]

Since \( (\beta + \frac{M}{\sigma}) \rho N < 1 \), in view of hypothesis (v) this leads to contradiction
which completes the proof of the lemma.

Define the matrix \( G(t) \) as
\[
G(t) = \int_{t_j}^{t^*} \psi^*(s, t) \psi(s, t) \, ds \text{ where } \psi^* \text{ is the transpose of } \psi.
\]
Clearly \( G(t) \) is symmetric. And define \( W(t,x) = \langle G(t)x,x \rangle \) and 

\[ V(t,x) = W(t,x) + \beta \int_{t_0}^{t} \int_{t_0}^{s} \|K(u,s)\| \|x(s)\| \, ds. \]

**Theorem 2.1:** Assume the following conditions hold.

(i) \( L\|x\| \leq \langle G(t)x,x \rangle \leq \frac{1}{2L}\|x\| \)

(ii) \( \|G(t)x\| \leq \hat{K} \langle G(t)x,x \rangle \)

(iii) \( -\hat{M} + \beta \int_{t}^{\infty} \|K(u,t)\| \, du \leq 0, \beta \geq \hat{K} \)

(iv) \( \|x\| > \|x + I_k(x)\| \) and

\[ \langle G(t)x,x \rangle > \frac{1}{2} \langle G(t)(x + I_k(x)),(x + I_k(x)) \rangle \]

where \( L, \hat{M}, \hat{K}, \) and \( \beta \) are positive real numbers.

Then the zero solution of (2.1) is uniformly stable.

**Proof:** Let \( W(t,x) = \langle G(t)x,x \rangle \)

\[ W'(t,x) = \frac{\langle G'(t)x,x \rangle}{2\langle G(t)x,x \rangle} + \frac{2G(t)x,x}{2\langle G(t)x,x \rangle} \]

we have

\[ \frac{\partial \psi(s,t)}{\partial t} = \psi(s,t)A(t) \]

\[ \frac{\partial \psi^T(s,t)}{\partial t} = \psi^T(s,t)A^T(t) \]

Hence

\[ G'(t) = -I - \int_{t_0}^{\infty} \left[ \frac{\partial \psi^T(s,t)}{\partial t} \psi(s,t) \psi^T(s,t) \frac{\partial \psi(s,t)}{\partial t} \right] \]

Which implies

\[ G'(t) = -I - A^T(t)G(t) - G(t)A(t) \]
Hence

\[ W'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t) x, x \rangle^{\frac{1}{2}}} + \frac{\langle G(t) x, \int_{t_0}^{t} K(t, s) x(s) \, ds \rangle}{\langle G(t) x, x \rangle^{\frac{1}{2}}} \]

for \( t \neq t_k, (t, x) \in U_{G_k} \)

Now

\[ V'(t, x) = \frac{-\langle x, x \rangle}{2\langle G(t) x, x \rangle^{\frac{1}{2}}} + \frac{\langle G(t) x, \int_{t_0}^{t} K(t, s) x(s) \, ds \rangle}{\langle G(t) x, x \rangle^{\frac{1}{2}}} \]

\[ + \beta \int_{t_0}^{t} \| K(u, t) \| \| u \| \| x(t) \| \, du \| x(s) \| \, ds \quad \text{for } t \neq t_k, (t, x) \in U_{G_k} \]

by (i) and (ii) we get

\[ V'(t, x) \leq -\hat{M} \| x \| + \hat{K} \int_{t_0}^{t} \| K(t, s) \| \| x(s) \| \, ds \]

(2.1)

\[ + \beta \int_{t_0}^{t} \| K(u, t) \| \| u \| \| x(t) \| \, du \| x(s) \| \, ds \]

Hence in view of assumption (iii) it follows that

\[ V'(t, x(s)) \leq 0 \quad \text{for } t \neq t_k, (t, x) \in U_{G_k} \]

(2.1)

This implies for \( t \neq t_k \) that by hypothesis (iv)

\[ \| x(t) \| \leq V(t, x) \leq V(t_0, x_0) \leq \mathcal{W}(t_0, x_0) \leq \frac{1}{2\hat{M}} \| x_0 \| \]

this gives the uniform stability of (2.1)

Remark 2.1: In the above theorem it is not assumed the descresent property on \( V \).
Theorem 2.2 Assume the following conditions hold for \|x\| < \rho

(i) \[L\|x\| \leq \langle G(t) x, x \rangle \leq \frac{1}{2} + \frac{1}{2M} \|x\|\]

(ii) \[\|G(t) x\| \leq \langle \hat{K} G(t) x, x \rangle \leq \frac{1}{2}\]

(iii) \[\gamma \leq \hat{M} - \beta \int_{t} \|K(u, t)\| du \text{ for some } \gamma > 0, \beta > \hat{K}\]

(iv) \[\|x\| > \|x + I_k(x)\| \text{ and } \langle G(t) x, x \rangle \leq \langle G(t) (x + I_k(x)), (x + I_k(x)) \rangle \leq \frac{1}{2}\]

where \(L, \hat{M}, \hat{K}, \gamma, \beta\) are positive real numbers.

Then the zero solution of (2.1) is uniformly asymptotically stable.

Proof: By Theorem 2.1 the zero solution of (2.1) is uniformly stable. Following the proof of Theorem 2.1 one obtains

\[\frac{d}{dt} V(t, x) \leq -\gamma \|x\| \text{ for } t \neq \tau_k, \|x\| < \rho \text{ and } (t, x) \in \bar{U}G_k.\]

(2.1)

Let \(s\) be the number corresponding to \(\epsilon\) in the definition of uniform stability.

Take \(T(\epsilon) = \left[\frac{1}{2M}\right] \|x_0\|\) where \(x(t_0) = x_0\)

We now claim that \(\|x(t^*, t_0, x_0)\| \leq \delta\) for some \(t^* \in [t_0, t_0 + \tau]\)

Whenever \(\|x(s)\| < \rho\) for \(0 \leq s \leq t_0\).

For if \(\|x(t, t_0, x_0)\| > \delta\) for all \(t \in [t_0, t_0 + \tau]\), then

By hypotheses (i) and (iv)

\[0 < L\delta = L\|x(t, t_0, x_0)\| \leq V(t, x(t)) \leq V(t, x_0) + \int_{t_0}^{t} \nu_2(s, x(s)) ds \leq \left[\frac{1}{2M}\right] \rho - \gamma \int_{t_0}^{t} \|x(s)\| ds\]
put \( t = t_0 + T \), then we get

\[
0 < L \delta \leq \left[ \frac{1}{2M} \right] \rho - T \eta \delta \\
\leq \left[ \frac{1}{2M} \right] \rho - \left[ \frac{1}{2M} \right] \frac{1}{\eta \delta} \rho \eta \delta = 0
\]

and thus we have a contradiction.

Hence there exists \( t^* \in [t_0, t_0 + T] \) such that \( \| x(t^*, t_0, x_0) \| < \delta \).

By uniform stability it follows that \( \| x(t, t_0, x_0) \| < \epsilon \) for all \( t > t^* \) or \( t \geq t_0 + T \) which completes the proof.

REFERENCES


