Research Article

Bayes’ Model of the Best-Choice Problem with Disorder

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We consider the best-choice problem with disorder and imperfect observation. The decision-maker observes sequentially a known number of i.i.d random variables from a known distribution with the object of choosing the largest. At the random time the distribution law of observations is changed. The random variables cannot be perfectly observed. Each time a random variable is sampled the decision-maker is informed only whether it is greater than or less than some level specified by him. The decision-maker can choose at most one of the observation. The optimal rule is derived in the class of Bayes’ strategies.

1. Introduction

In the papers we consider the following best-choice problem with disorder and imperfect observations. A decision-maker observes sequentially \( n \) iid random variables \( \xi_1, \ldots, \xi_{\theta-1}, \xi_\theta, \ldots, \xi_n \). The observations \( \xi_1, \ldots, \xi_{\theta-1} \) are from a continuous distribution law \( F_1(x) \) (state \( S_1 \)). At the random time \( \theta \), the distribution law of observations is changed to continuous distribution function \( F_2(x) \) (i.e., the disorder happen—state \( S_2 \)). The moment of the disorder has a geometric distribution with parameter \( 1 - \alpha \). The observer knows parameters \( \alpha, F_1(x), \) and \( F_2(x) \), but the exact moment \( \theta \) is unknown.

At each time in which a random variable is sampled, the observer has to make a decision to accept (and stop the observation process) or reject the observation (and continue the observation process). If the decision-maker decided to accept at step \( k \) \( (1 \leq k \leq n) \), she receives as the payoff the value of the random variable discounted by the factor \( \lambda^{k-1} \), where \( 0 < \lambda < 1 \). The random variables cannot be perfectly observed. The decision-maker is only informed whether the observation is greater than or less than some level specified by her.
The aim of the decision-maker is to maximize the expected value of the accepted discounted observation.

We find the solution in the class of the following strategies. At each moment \( k \) (1 \( \leq \) \( k \) \( \leq \) \( n \)), the observer estimates the \textit{a posteriori} probability of the current state and specifies the threshold \( s = s_{n-k} \). The decision-maker accepts the observation \( x_k \) if and only if it is greater than the corresponding threshold \( s \).

This problem is the generalization of the best-choice problem [1, 2] and the quickest determination of the change-point (disorder) problem [3–5]. The best-choice problems with imperfect information were treated in [6–8]. Only few papers related to the combined best-choice and disorder problem are published [9–11]. Yoshida [9] considered the full-information case and found the optimal stopping rule which maximizes the probability that accepted value is the largest of all \( \theta + m - 1 \) random variables for a given integer \( m \). Closely related work to this study is Sakaguchi [10] where the optimality equation for the optimal expected reward is derived for the full-information model. In [11], we constructed the solution of the combined best-choice and disorder problem in the class of single-level strategies, and, in this paper, we search the Bayes’ strategy which maximizes the expected reward in the model with imperfect observation.

\section{2. Optimal Strategy}

According to the problem the observer does not know the current state (\( S_1 \) or \( S_2 \)). But she can estimate the state using the Bayes’ formula:

\[
\pi_s = \pi(s) = P[S_1 \mid x \leq s] = \frac{P(S_1)P(x \leq s \mid S_1)}{P(x \leq s)} = \frac{\alpha \pi F_1(s)}{F_\pi(s)}, \tag{2.1}
\]

Here, \( s = s_i \) is the threshold specified by the decision-maker within \( i \) steps until the end (i.e., at the step \( n - i \)), \( \pi \) is the \textit{a priori} probability of the state \( S_1 \) (i.e., before getting the information that \( x \leq s \)), \( F_\pi(s) = \pi F_1(s) + \pi F_2(s) \), and \( \pi = 1 - \pi \).

We use the dynamic programming approach to derive the optimal strategy. Let \( v_i(\pi) \) be the payoff that the observer expects to receive using the optimal strategy within \( s \) steps until the end. The optimality equation is as follows:

\[
v_i(\pi) = \max_s E[\lambda v_{i-1}(\pi_s)I_{x \leq s} + x I_{x > s}], \hspace{1cm} i \geq 1, \hspace{1cm} v_0(\pi) = 0 \hspace{1cm} \forall \pi. \tag{2.2}
\]

Simplifying (2.2), we get

\[
v_i(\pi) = \max_s [\lambda v_{i-1}(\pi_s)F_\pi(s) + \pi E_1(s) + \pi F_2(s)], \hspace{1cm} i \geq 1, \hspace{1cm} v_0(\pi) = 0 \hspace{1cm} \forall \pi. \tag{2.3}
\]

Here, \( E_k(s) = \int_s^\infty x dF_k(x), k = 1, 2 \).

The following theorem gives the presentation of the expected payoff in linear form on \( \pi \).
Lemma 2.2. Assuming

Theorem 2.1. For any i the function \( v_i(\pi) \) can be written in the form

\[
v_i(\pi) = \pi A_i(s_1, \ldots, s_i) + B_i(s_1, \ldots, s_i),
\]

where

\[
s_i = s_i(\pi) = \arg \max_s [\lambda v_{i-1}(\pi_s) F_s(s) + \pi E_1(s) + \overline{E}_2(s)], \quad i \geq 1, \ 0 \leq \pi \leq 1. \tag{2.5}
\]

Proof. Using the formula (2.3), one can show that

\[
v_1(\pi) = \max_s [\pi (E_1(s) - E_2(s)) + E_2(s)] = \pi A_1(s_1) + B_1(s_1),
\]

where \( A_1(s_1) = E_1(s_1) - E_2(s_1) \), \( B_1 = E_2(s_1) \) and

\[
s_1 = s_1(\pi) = \arg \max_s [\pi (E_1(s) - E_2(s)) + E_2(s)], \quad 0 \leq \pi \leq 1. \tag{2.7}
\]

Threshold \( s_1 = s_1(\pi) \) is the solution of (2.3) for \( 0 \leq \pi \leq 1 \) for \( i = 1 \).

Assume the theorem is correct for certain \( i = k \). Then, for \( i = k + 1 \)

\[
v_{k+1}(\pi) = \max_s [\lambda (\pi A_k(s_1, \ldots, s_k) + B_k(s_1, \ldots, s_k)) F_s(s) + \pi E_1(s) + \overline{E}_2(s)]
\]

\[
= \max_s [\pi (\lambda F_1(s) A_k(s_1, \ldots, s_k) + B_k(s_1, \ldots, s_k) (F_1(s) - F_2(s)) + E_1(s) - E_2(s))
\]

\[
+ \lambda B_k(s_1, \ldots, s_k) F_2(s) + E_2(s)]
\]

\[
= \pi A_{k+1}(s_1, \ldots, s_{k+1}) + B_{k+1}(s_1, \ldots, s_{k+1}),
\]

where

\[
A_{k+1}(s_1, \ldots, s_{k+1}) = \lambda A_1(s_1, \ldots, s_k) A_k(s_1, \ldots, s_k) + \lambda B_k(s_1, \ldots, s_k) (F_1(s) - F_2(s)) + E_1(s) - E_2(s),
\]

\[
B_{k+1}(s_1, \ldots, s_{k+1}) = \lambda B_k(s_1, \ldots, s_k) F_2(s) + E_2(s),
\]

\[
s_i = s_i(\pi) = \arg \max_s [\lambda v_{i-1}(\pi_s) F_s(s) + \pi E_1(s) + \overline{E}_2(s)], \quad i \geq 1, \ 0 \leq \pi \leq 1. \tag{2.9}
\]

The theorem is proved. \( \square \)

The following lemma takes place.

Lemma 2.2. Assuming \( E_k < \infty, k = 1,2 \) as \( i \to \infty \), there is a limit of the expected payoff \( v_i(\pi) \to v(\pi) \).

Proof. It is obvious that the sequence \( v_i(\pi) \) is increasing by \( i \).
Now, we prove that the sequence of the expected payoffs has an upper bound.

\[ v_1(\pi) \leq \pi E_1 + \pi E_2, \]

\[ E_k = \int_{0}^{\infty} x dF_k(x), \quad k = 1, 2 \]  \hspace{1cm} (2.10)

\[ v_2(\pi) = \max_s [\lambda v_1(\pi_s) F_x(s) + \pi E_1(s) + \pi E_2(s)] \]

\[ \leq \lambda (\pi E_1 + \pi E_2) + \pi E_1 + \pi E_2. \]

Further one can show using the induction that for any \( i \geq 1 \) and any \( 0 \leq \pi \leq 1 \) the expected payoff at the step \( i \) has the upper bound

\[ v_i(\pi) \leq \frac{\pi E_1 + \pi E_2}{1 - \lambda}. \]  \hspace{1cm} (2.11)

The lemma is proved. \hfill \Box

**Corollary 2.3.** *Theorem 2.1 and the lemma yield that there are such \( A \) and \( B \) that*

\[ \lim_{i \to \infty} v_i(\pi) = \lim_{i \to \infty} (\pi A_i(s_1, \ldots, s_i) + B_i(s_1, \ldots, s_i)) = \pi A + B = v(\pi). \]  \hspace{1cm} (2.12)

As \( i \to \infty \) the expected payoff satisfies the following equation:

\[ v(\pi) = \lim_{i} v_i(\pi) = \max_s [\lambda v(\pi_s) F_x(s) + \pi E_1(s) + \pi E_2(s)]. \]  \hspace{1cm} (2.13)

To find the components of the expected payoff for a case of huge number of observation we should solve the following equation:

\[ \pi A + B = \max_s [\pi (\lambda a F_1(s) A + \lambda B(F_1(s) - F_2(s)) + E_1(s) - E_2(s)) + \lambda BF_2(s) + E_2(s)], \]  \hspace{1cm} (2.14)

therefore,

\[ A = \lambda a F_1(s) A + \lambda B(F_1(s) - F_2(s)) + E_1(s) - E_2(s), \]

\[ B = \lambda BF_2(s) + E_2(s). \]  \hspace{1cm} (2.15)

The solution of the system is as follows

\[ A = \frac{E_1(s)(1 - \lambda F_2(s)) - E_2(s)(1 - \lambda F_1(s))}{(1 - \lambda F_2(s))(1 - \lambda a F_1(s))}, \]

\[ B = \frac{E_2(s)}{1 - \lambda F_2(s)}. \]  \hspace{1cm} (2.16)
The expected payoff is

$$v(\pi) = \max_s (\pi A + B) \quad (2.17)$$

and the optimal threshold is

$$s = s(\pi) = \arg \max_s (\pi A + B). \quad (2.18)$$

The above results are summarized in the following theorem.

**Theorem 2.4.** For $i \to \infty$, the solution of (2.3) is defined as

$$v(\pi) = \max_s (\pi A + B), \quad (2.19)$$

where

$$s = s(\pi) = \arg \max_s (\pi A + B),$$

$$A = \frac{E_1(s)(1 - \lambda F_2(s)) - E_2(s)(1 - \lambda F_1(s))}{(1 - \lambda F_2(s))(1 - \lambda F_1(s))}, \quad (2.20)$$

$$B = \frac{E_2(s)}{1 - \lambda F_2(s)}.$$

### 3. Examples

Consider the examples of using the Bayes’ strategy $B$ defined by the formula (2.18) comparing with two strategies with constant thresholds that do not depend on $\pi$.

#### 3.1. Normal Distribution

Consider the example of the normal distribution of the random variables where functions $F_1(x)$ and $F_2(x)$ have the variance $\sigma^2 = 1$ and the expectation $\mu_1 = 10$ and $\mu_2 = 9$, respectively.

Strategies $A_1$ and $A_2$ with constant thresholds defined by the following formula:

$$s = \frac{E(s)}{1 - \lambda F(s)}, \quad (3.1)$$

where $F(s) \equiv F_1(s)$ and $E(s) \equiv E_1(s)$ for the strategy $A_1$; $F(s) \equiv F_2(s)$ and $E(s) \equiv E_2(s)$ for the strategy $A_2$.

The values of the thresholds of strategies $A_1$ and $A_2$ depending on discount rate are tabulated in Table 1.

Table 1 shows how much the discount rate is affect on the thresholds.

Figure 1 shows the graphics of the optimal thresholds for strategies $A_1$ and $A_2$ ($s_1$ and $s_2$, resp.) and strategy $B$ ($s_{opt}$) depending on $\pi$. As the figure shows, the strategy $B$ depends
Table 1: The values of the thresholds of strategies $A_1$ and $A_2$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Strategy $A_1$</th>
<th>Strategy $A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>10.851</td>
<td>9.902</td>
</tr>
<tr>
<td>0.9</td>
<td>9.088</td>
<td>8.210</td>
</tr>
<tr>
<td>0.7</td>
<td>7.000</td>
<td>6.300</td>
</tr>
</tbody>
</table>

Figure 1: Graphics of the optimal thresholds for strategies $A_1$, $A_2$, and $B$ for $\alpha = 0.9$, $\lambda = 0.99$.

Table 2: Main characteristics of the best-choice process for $\alpha = 0.9$, $\lambda = 0.99$.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Strategy $A_1$</th>
<th>Strategy $A_2$</th>
<th>Strategy $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected payoff</td>
<td>10.035</td>
<td>10.429</td>
<td>10.500</td>
</tr>
<tr>
<td>Average time of accepting the observation</td>
<td>14.526</td>
<td>2.472</td>
<td>3.072</td>
</tr>
<tr>
<td>Average number of steps after the disorder</td>
<td>30.406</td>
<td>4.503</td>
<td>5.031</td>
</tr>
<tr>
<td>Number of the values accepted before the disorder, $%$</td>
<td>64.100</td>
<td>83.066</td>
<td>79.738</td>
</tr>
</tbody>
</table>

on the a posterior probability of the state $S_1(\pi)$. As $\pi$ tends to zero, the optimal threshold of the strategy $B$ tends to threshold $s_2$.

We compare the payoffs that the observer expects to receive using different strategies. Define $V_\pi$ as the expected payoff for $\pi = 1$ and depending on probability of disorder $\alpha$.

Figure 2 shows the numerical results of the expected payoffs of the observer who uses the strategies $A_1$, $A_2$, and $B$ (thresholds $s_1$, $s_2$, and $s_{opt}$, resp.).

The expected payoff of the observer who uses the Bayes’ strategy $B$ is greater if she uses one of the strategies $A_1$ or $A_2$. The difference is significant for $\alpha \in [0.75, 0.98]$, because of uncertainty of the current state of the system.

Table 2 shows the numerical results of the main characteristics of the best-choice process.

For the small probability of the disorder ($1-\alpha = 0.1$), the expected payoff according to the strategy $A_2$ is greater (10.429) than according to the strategy $A_1$ (10.035). But the Bayes’ strategy $B$ that depends on $\pi$ gives the largest expected payoff (10.500).

Table 2 shows that the average time of accepting the observation is increasing with respect to the value of the threshold. Note that the strategy $A_1$ does not depend on the disorder and this leads to a high value of the average time of accepting the observation. Both strategies $A_2$ and $B$ have a small average time of accepting the observation.
Figure 2: Expected payoffs of the observer who uses the strategies \( A_1, A_2 \) and \( B \) for \( \alpha = 0.9, \lambda = 0.99 \).

\[
\begin{array}{|c|c|c|}
\hline
\lambda & \text{Strategy } A_1 & \text{Strategy } A_2 \\
\hline
0.99 & 6.756 & 3.378 \\
0.9 & 3.358 & 1.679 \\
\hline
\end{array}
\]

Table 3: The values of the thresholds of strategies \( A_1 \) and \( A_2 \).

Table 4: Main characteristics of the best-choice process for \( \alpha = 0.9, \lambda = 0.99 \).

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Strategy ( A_1 )</th>
<th>Strategy ( A_2 )</th>
<th>Strategy ( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected payoff</td>
<td>2.355</td>
<td>4.438</td>
<td>4.499</td>
</tr>
<tr>
<td>Average time of accepting the observation</td>
<td>678.930</td>
<td>15.397</td>
<td>16.923</td>
</tr>
<tr>
<td>Average number of steps after the disorder</td>
<td>856.535</td>
<td>29.110</td>
<td>29.610</td>
</tr>
<tr>
<td>Number of the values accepted before the disorder, %</td>
<td>21.57</td>
<td>70.89</td>
<td>56.01</td>
</tr>
</tbody>
</table>

3.2. Exponential Distribution

Consider the example of the exponential distribution of the observations. Let \( F_1(x) \) and \( F_2(x) \) have the exponential distribution with parameters \( \lambda_1 = 0.5 \) and \( \lambda_2 = 1 \), respectively. As in the previous example, consider the strategies \( A_1 \) and \( A_2 \) comparing with the Bayes’ strategy \( B \),

\[
s = \frac{E(s)}{1 - \lambda F(s)},
\]

where \( F(s) \equiv F_1(s) \) and \( E(s) \equiv E_1(s) \) for the strategy \( A_1 \); \( F(s) \equiv F_2(s) \) and \( E(s) \equiv E_2(s) \) for the strategy \( A_2 \).

Table 3 shows the values of the thresholds for the strategies \( A_1 \) and \( A_2 \) depending on the discount rate.

The value of the optimal threshold of the strategy \( B \) as in the case of the normal distribution of the observations is increasing by \( \pi \) and equal to the threshold of the strategy \( A_2 \) at \( \pi = 0 \). The graphics of the expected payoffs have the same view as in Figure 2. Table 4 shows the main characteristics of the best-choice process for different strategies.

As in the previous example, the Bayes’ strategy gives better payoff than the strategy \( A_2 \), but it has bigger average time of accepting the observation. The strategy \( A_1 \) is the worst for all the parameters.
4. Results

In the article, we consider the best-choice problem with disorder and imperfect observations. We propose the Bayes’ strategy where the threshold depends on the \textit{a posteriori} probability of the disorder. The numerical results show that this strategy gives better expected payoff than the constant strategies.

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References

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