Research Article
On the Lower Bound for the Number of Real Roots of a Random Algebraic Equation
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We estimate a lower bound for the number of real roots of a random algebraic equation whose random coefficients are dependent normal random variables.

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1. Introduction
Let \( N_n(R, \omega) \) be the number of real roots of the random algebraic equation
\[
F_n(x, \omega) = \sum_{\nu=0}^{n} a_{\nu}(\omega)x^\nu = 0, \tag{1.1}
\]
where the \( a_{\nu}(\omega), \nu = 0, 1, \ldots, n \), are random variables defined on a fixed probability space \((\Omega, \mathcal{A}, \Pr)\) assuming real values only.

During the past 40–50 years, the majority of published researches on random algebraic polynomials has concerned the estimation of \( N_n(R, \omega) \). Works by Littlewood and Offord [1], Samal [2], Evans [3], and Samal and Mishra [4–6] in the main concerned cases in which the random coefficients \( a_{\nu}(\omega) \) are independent and identically distributed.

For dependent coefficients, Sambandham [7] considered the upper bound for \( N_n(R, \omega) \) in the case when the \( a_{\nu}(\omega), \nu = 0, 1, \ldots, n \), are normally distributed with mean zero and joint density function
\[
|M|^{1/2}(2\pi)^{-(n+1)/2} \exp \left(-\frac{1}{2}a'Ma\right), \tag{1.2}
\]
where \( M^{-1} \) is the moment matrix with \( \sigma_i = 1, \rho_{ij} = \rho, \ \text{if} \ \rho < 1, \ i \neq j, \ i, j = 0, 1, \ldots, n \) and \( a' \) is the transpose of the column vector \( a \). Also, Uno and Negishi [8] obtained the same result as Sambandham in the case of the moment matrix with \( \sigma_i = 1, \rho_{ij} = \rho_{|i-j|} \).
(i ≠ j), i, j = 0, 1, . . . , n, where ρj is a nonnegative decreasing sequence satisfying ρ1 < 1/2 and \( \sum_{j=1}^{\infty} \rho_j < \infty \) in (1.2).

The lower bound for \( N_n(R, \omega) \) in the case of dependent normally distributed coefficients was estimated by Renganathan and Sambandham [9] and Nayak and Mohanty [10] under the same condition of Sambandham [7]. Uno [11] pointed out the defect in the proofs of the above papers and obtained the result for the lower bound. Additionally, Uno [12] estimated the strong result for this particular problem in the sense of Evans [3]. The term strong indicates that the estimation for the exceptional set is independent of the degree n.

The object of this paper is to find the lower bound for \( N_n(R, \omega) \) when the coefficients are nonidentically distributed dependent normal random variables. We remark that this result is the general form of Uno [11] and that the exceptional set is dependent on the degree n. In this paper, we suppose that the \( a_\nu(\omega), \nu = 0, 1, \ldots, n \), have mean zero, and the moment with

\[
\rho_{ij} = \begin{cases} 
1 & (i = j), \\
\rho_{|i-j|} & (1 \leq |i-j| \leq m), \\
0 & (|i-j| > m),
\end{cases}
\]

for a positive integer m, where \( 0 \leq \rho_j < 1, \, j = 1, 2, \ldots, m \) in (1.2). That is to say we assume the \( a_\nu(\omega) \)'s to be m-dependent stationary Gaussian random variables. With Yoshihara ([13, page 29]), we see that this assumption is equivalent to the following two statements for a stationary Gaussian sequence:

(i) \( \{a_\nu\} \) is \( \ast \)-mixing;
(ii) \( \{a_\nu\} \) is \( \phi \)-mixing.

Throughout the paper, we suppose \( n \) is sufficiently large. We will follow the line of proof of Samal and Mishra [5].

**Theorem 1.1.** Let

\[
f_n(x, \omega) = \sum_{\nu=0}^{n} a_\nu(\omega) b_\nu x^\nu = 0
\]

be a random algebraic equation of degree n, where the \( a_\nu(\omega) \)'s are dependent normally distributed with mean zero, and the moment matrix given by (1.3) and the \( b_\nu, \nu = 0, 1, \ldots, n \), be positive numbers such that \( \lim_{n \to \infty} (k_n/t_n) \) is finite, where \( k_n = \max_{0 \leq \nu \leq n} b_\nu \) and \( t_n = \min_{0 \leq \nu \leq n} b_\nu \).

Then for \( n > n_0 \), the number of real roots of most of the equations \( f_n(x, \omega) = 0 \) is at least \( \varepsilon_n \log n \) outside a set of measure at most

\[
\frac{\mu}{\varepsilon_n \log n} + \left( \frac{k_n}{t_n} \right)^\beta \exp \left( -\frac{\mu' \beta}{\varepsilon_n} \right), \quad \beta > 0,
\]

provided \( \varepsilon_n \) tends to zero, but \( \varepsilon_n \log n \) tends to infinity as \( n \) tends to infinity, and \( \mu \) and \( \mu' \) are positive constants.
2. Proof of theorem

Let \( \{\lambda_n\} \) be any sequence tending to infinity as \( n \) tends to infinity and \( M \) is the integer defined by

\[
M = \left[ \alpha^2 \lambda_n^2 \left( \frac{k_n}{t_n} \right)^2 \right] + 1, \tag{2.1}
\]

where \( \alpha \) is a positive constant and \([x]\) denotes the greatest integer not exceeding \( x \). Let \( k \) be the integer determined by

\[
M^{2k} \leq n < M^{2k+2}. \tag{2.2}
\]

We will consider \( f_n(x, \omega) \) at the points

\[
x_l = \left( 1 - \frac{1}{M^{2l}} \right)^{1/2} \tag{2.3}
\]

for \( l = [k/2] + 1, [k/2] + 2, \ldots, k \).

Let

\[
f_n(x_l, \omega) = \sum_1 a_\nu(\omega) b_\nu x_l^\nu + \left( \sum_2 + \sum_3 \right) a_\nu(\omega) b_\nu x_l^\nu = U_l(\omega) + R_l(\omega), \quad \text{(say),} \tag{2.4}
\]

where \( \nu \) ranges from \( M^{2l-1} + 1 \) to \( M^{2l+1} \) in \( \sum_1 \), from 0 to \( M^{2l-1} \) in \( \sum_2 \) and from \( M^{2l+1} + 1 \) to \( n \) in \( \sum_3 \).

The following lemmas are necessary for the proof of the theorem. We will use the fact that each \( a_\nu(\omega) \) has marginal frequency function \((2\pi)^{-1/2} \exp(-u^2/2)\).

**Lemma 2.1.** For \( \alpha_1 > 0 \),

\[
\sigma_l > \alpha_1 t_n M^l, \tag{2.5}
\]

where

\[
\sigma_l^2 = \sum_{i=M^{2l-1}+1}^{M^{2l+1}} b_i^2 x_l^{2i} + 2 \sum_{i=M^{2l-1}+1}^{M^{2l+1}} \sum_{j=i+1}^{M^{2l+1}} b_i b_j x_l^{i+j} \rho_{j-i}, \tag{2.6}
\]

**Proof.** First, we have

\[
\sum_{i=M^{2l-1}+1}^{M^{2l+1}} b_i^2 x_l^{2i} > t_n^2 \sum_{i=M^{2l-1}+1}^{M^{2l}} x_l^{2i} > \left( \frac{B}{A} \right) t_n^2 M^{2l}, \tag{2.7}
\]

where \( A \) and \( B \) are positive constants such that \( A > 1 \) and \( 0 < B < 1 \).
Second, we get

\[
M^{2l+1} - 1 \sum_{i=M^{2l+1} + 1}^{M^{2l+1}} \sum_{j=M^{2l+1} + 1}^{M^{2l+1}} b_i b_j x_{l}^{i+j} \rho_{j-i} > t_n^2 \sum_{i=M^{2l+1} + 1}^{M^{2l+1}} \sum_{j=M^{2l+1} + 1}^{M^{2l+1}} x_{l}^{i+j} \rho_{j-i} \]

\[
= t_n^2 \frac{x_{l}^{2(M^{2l+1} + 1)}}{1 - x_{l}^{2}} \left\{ \sum_{i=1}^{m} \rho_i x_{l}^i - \sum_{i=1}^{m} \rho_i x_{l}^{2(M^{2l+1} - M^{2l+1}) - i} \right\} \geq \left( \frac{B'}{A'} \right) \rho_0 t_n^2 M^{2l},
\]

where \( \rho_0 = \sum_{j=1}^{m} \rho_j \) and \( A' \) and \( B' \) are positive constants satisfying \( A' > 1 \) and \( 0 < B' < 1 \).

So we get

\[
\sigma_{l}^2 \geq \alpha_1^2 t_n^2 M^{2l},
\]

where \( \alpha_1 \) is a positive constant, as required.

\[\square\]

Lemma 2.2. Let

\[
\Pr \left( \left\{ \omega; \left| \sum_{2} a_\nu(\omega) b_\nu x_{l}^{\nu} \right| > \lambda_n \tilde{\sigma}_l \right\} \right) < \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^2/2}}{\lambda_n},
\]

where

\[
\tilde{\sigma}_l^2 = \sum_{i=0}^{M^{2l+1} - 1} b_i^2 x_{l}^{2i} + 2 \sum_{i=0}^{M^{2l+1} - 1} \sum_{j=i+1}^{M^{2l+1}} b_i b_j x_{l}^{i+j} \rho_{j-i}.
\]

Proof. We get

\[
\Pr \left( \left\{ \omega; \left| \sum_{2} a_\nu(\omega) b_\nu x_{l}^{\nu} \right| > \lambda_n \tilde{\sigma}_l \right\} \right) = \sqrt{\frac{2}{\pi}} \int_{\lambda_n}^{\infty} e^{-u^2/2} du < \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^2/2}}{\lambda_n}.
\]

\[\square\]

Lemma 2.3. Let

\[
\Pr \left( \left\{ \omega; \left| \sum_{3} a_\nu(\omega) b_\nu x_{l}^{\nu} \right| > \lambda_n \tilde{\sigma}_l \right\} \right) < \sqrt{\frac{2}{\pi}} \frac{e^{-\lambda_n^2/2}}{\lambda_n},
\]

where

\[
\tilde{\sigma}_l^2 = \sum_{i=M^{2l+1} + 1}^{n} b_i^2 x_{l}^{2i} + 2 \sum_{i=M^{2l+1} + 1}^{n-1} \sum_{j=i+1}^{n} b_i b_j x_{l}^{i+j} \rho_{j-i}.
\]

The proof is similar to that of Lemma 2.2.

Lemma 2.4. For a fixed \( l \),

\[
\Pr \left( \left\{ \omega; \left| R_l(\omega) \right| < \sigma^2 \right\} \right) > 1 - 2 \sqrt{\frac{2}{\pi}} \frac{1}{\lambda_n} e^{-\lambda_n^2/2}.
\]
Proof. By Lemmas 2.2 and 2.3, we get, for a given \( l \),
\[
\left| R_l(\omega) \right| < \lambda_n(\bar{\sigma}_l + \tilde{\sigma}_l) \tag{2.16}
\]
outside a set of measure at most \( 2(2/\pi)^{1/2}\lambda_n^{-1}\exp(-\lambda_n^2/2) \). Again, we have
\[
\sum_{i=0}^{M^{2l-1}} b_i^2 x_i^{2l} \leq 2k_n^2 M^{2l-1},
\]
\[
\sum_{i=0}^{M^{2l-1}-1} \sum_{j=i+1}^{M^{2l-1}} b_i b_j x_i^{i+j+1} \rho_{j-i} \leq k_n^2 \sum_{i=1}^{M^{2l-1}-(i-1)} x_i^{2j+i+2} \leq \rho_0 k_n^2 M^{2l-1}. \tag{2.17}
\]
Hence we get, for a positive constant \( \alpha_2 \),
\[
\tilde{\sigma}_l^2 \leq \alpha_2^2 k_n^2 M^{2l-1}. \tag{2.18}
\]
Similarly, we have
\[
\tilde{\tilde{\sigma}}_l^2 \leq \alpha_3^2 k_n^2 M^{2l-1} \tag{2.19}
\]
for a positive constant \( \alpha_3 \). Therefore, we obtain, outside the exceptional set,
\[
\left| R_l(\omega) \right| < \lambda_n (\alpha_2 + \alpha_3) k_n M^{1-(1/2)} \leq \left( \frac{\alpha_2 + \alpha_3}{\alpha_1} \frac{k_n}{\lambda_n \sigma_l} \right) M^{1/2} < \sigma_l, \tag{2.20}
\]
by Lemma 2.1 and (2.1).

Let us define random events \( E_p, F_p \) by
\[
E_p = \{ \omega; U_{3p}(\omega) \geq \sigma_{3p}, U_{3p+1}(\omega) < -\sigma_{3p+1} \},
\]
\[
F_p = \{ \omega; U_{3p}(\omega) < -\sigma_{3p}, U_{3p+1}(\omega) \geq \sigma_{3p+1} \}. \tag{2.21}
\]
It can be easily seen that
\[
\Pr (E_p \cup F_p) = \delta_p \text{ (say) > } \delta, \tag{2.22}
\]
where \( \delta > 0 \) is a certain constant. Let \( \eta_p \) be a random variable such that
\[
\eta_p = \begin{cases} 
1 & \text{on } E_p \cup F_p, \\
0 & \text{elsewhere.} 
\end{cases} \tag{2.23}
\]
Then we get
\[
E(\eta_p) = \delta_p, \quad V(\eta_p) = \delta_p - \delta_p^2. \tag{2.24}
\]
Let \( q \) be the total number of pairs \((U_{3p}, U_{3p+1})\) for which
\[
\left[ \frac{k}{2} \right] + 1 \leq 3p < 3p + 1 \leq k, \tag{2.25}
\]
q must be at least equal to \([k/3] - [([k/2] + 1)/3] - 1\). Take

$$\eta = \sum \eta_p,$$

where the summation is taken over all the \(q\) pairs. Applying Tschebyscheff inequality, we have, for \(0 < \varepsilon < \delta\),

$$\Pr \left( \left| \eta - E(\eta) \right| \geq q\varepsilon \right) \leq \frac{V(\eta)}{q^2\varepsilon^2} \leq \frac{\sum \delta_p}{q^2\varepsilon^2} \leq \frac{1}{q^2\varepsilon^2},$$

since for \(n\) sufficiently large, \(\text{Cov}(\eta_i, \eta_j) = 0\) \((i \neq j)\). But

$$q \geq \left[\frac{k}{3}\right] - \left[\frac{[k/2] + 1}{3}\right] - 1 \geq \frac{k}{3} - 1 - \left(\frac{k/2 + 1}{3}\right) - 1 = \frac{1}{6}(k - 14) \geq \mu_1 k,$$

where \(\mu_1\) is a positive constant. Therefore, outside a set of measure at most \(\mu_2/k\),

$$\left| \eta - E(\eta) \right| < q\varepsilon,$$

that is,

$$\eta - E(\eta) > -q\varepsilon$$

or

$$\eta > E(\eta) - q\varepsilon = \sum \delta_p - q\varepsilon > q(\delta - \varepsilon) \geq \mu_3 k,$$

where \(\mu_2\) and \(\mu_3\) are positive constants. Thus we have proved that outside a set of measure at most \(\mu_2/k\), either \(U_{3p} \geq \sigma_{3p}\) and \(U_{3p+1} < -\sigma_{3p+1}\), or \(U_{3p} < -\sigma_{3p}\) and \(U_{3p+1} \geq \sigma_{3p+1}\) for at least \(\mu_3 k\) values of \(l\).

Define

$$\xi_p = \begin{cases} 0 & \text{if } |R_{3p}| < \sigma_{3p}, \ |R_{3p+1}| < \sigma_{3p+1}, \\ 1 & \text{elsewhere}. \end{cases}$$

Let \(\xi_p = \eta_p - \eta_p \xi_p\). If \(\xi_p = 1\), there is a root of the polynomial in the interval \((x_{3p}, x_{3p+1})\). Hence the number of real roots in the interval \((x_{[k/2]+1}, x_k)\) must exceed \(\sum \xi_p\), where the summation is taken over all the \(q\) pairs. Now, by using Lemma 2.4, we have

$$E\left(\sum \eta_p \xi_p\right) = \sum E(\eta_p \xi_p) \leq \sum E(\xi_p) = \sum \Pr(\xi_p = 1)$$

$$\leq \sum \left\{ \Pr\left( |R_{3p}| \geq \sigma_{3p}\right) + \Pr\left( |R_{3p+1}| \geq \sigma_{3p+1}\right) \right\}$$

$$< \mu_4(k + 1) \frac{1}{\lambda_n} e^{-\lambda_n^2/2},$$

where \(\mu_4\) is a constant. Hence we have, for \(\beta > 0\),

$$\Pr\left( \left\{ \sum \eta_p \xi_p > \mu_4(k + 1) \frac{1}{\lambda_n} e^{-\lambda_n^2/2} \right\} \right) < \frac{E(\sum \eta_p \xi_p)}{\mu_4(k + 1) \lambda_n^{\beta-1} e^{-\lambda_n^2/2}} < \frac{1}{\lambda_n^\beta}.$$
So we get

\[ \sum \eta_p \zeta_p \leq \mu_4 (k+1) \lambda_n^{k+1} e^{-\lambda_n/2}, \] (2.35)

except for a set of measure at most $1/\lambda_n^\beta$. Therefore, we have, outside a set of measure at most $\mu_2/k + 1/\lambda_n^\beta$,

\[ N_n > \sum \xi_p > \mu_3 k - \mu_4 (k+1) \lambda_n^{k+1} e^{-\lambda_n/2} \geq k (\mu_3 - \epsilon_1), \] (2.36)

where $0 < \epsilon_1 < \mu_3$ (since $\mu_4 \lambda_n^{k+1} \exp(-\lambda_n/2)$ tends to zero as $n$ tends to infinity). But it follows from (2.1) and (2.2) that

\[ \mu_5 \left( \frac{k_n}{t_n} \right)^2 \lambda_n^2 \leq M \leq \mu_6 \left( \frac{k_n}{t_n} \right)^2 \lambda_n^2, \]
\[ \frac{\mu_7 \log n}{\log ((k_n/t_n) \lambda_n)} \leq k \leq \frac{\mu_8 \log n}{\log ((k_n/t_n) \lambda_n)}, \] (2.37)

where $\mu_i$, $i = 5, 6, 7, 8$, are constants. Hence we get outside the exceptional set

\[ N_n > \frac{\mu_9 \log n}{\log ((k_n/t_n) \lambda_n)}, \] (2.38)

where $\mu_9$ is a constant.

Taking $\lambda_n = (t_n/k_n) \exp (\mu_9/\epsilon_n)$, we obtain

\[ N_n > \epsilon_n \log n \] (2.39)

outside a set of measure at most

\[ \frac{\mu}{\epsilon_n \log n} + \left( \frac{k_n}{t_n} \right)^\beta \exp \left( -\frac{\mu'}{\epsilon_n} \right), \] (2.40)

where $\mu$ and $\mu'$ are constants. This completes the proof of the theorem.

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