For a $d$-dimensional array of random elements $\{V_n, n \in \mathbb{Z}_d^d\}$ in a real separable stable type $p$ ($1 \leq p < 2$) Banach space, a mean convergence theorem is established. Moreover, the conditions for the convergence in mean of order $p$ are shown to completely characterize stable-type $p$ Banach spaces.

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1. Introduction

Let $\mathbb{Z}_d^d$, where $d$ is a positive integer, denote the positive integer $d$-dimensional lattice points. The notation $m \prec n$, where $m = (m_1, m_2, \ldots, m_d)$ and $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d$, means that $m_i \leq n_i$, $1 \leq i \leq d$, $|n|$ is used for $\prod_{i=1}^d n_i$.

Gut [5] proved that if $\{X, X_n, n \in \mathbb{Z}_d^d\}$ is a $d$-dimensional array of i.i.d. random variables with $E|X|^p < \infty$ ($0 < p < 2$) and $EX = 0$ if $1 \leq p < 2$, then

$$\frac{\sum_{j \prec n} X_j}{|n|^{1/p}} \rightarrow 0 \quad \text{in } L^p \quad \text{as} \quad \min_{1 \leq i \leq d} n_i \rightarrow \infty,$$

(1.1)

where $(n_1, n_2, \ldots, n_d) = n \in \mathbb{Z}_+^d$.

Recently, Thanh [11] proved (1.1) under condition of uniform integrability of $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$.

Mean convergence theorems for sums of random elements Banach-valued are studied by many authors. The reader may refer to Wei and Taylor [12], Adler et al. [2], Rosalsky and Sreehari [9], or more recently, Rosalsky et al. [10], Cabrera and Volodin [3]. However, we are unaware of any literature of investigation on the mean convergence for multidimensional arrays of random elements in Banach spaces.

Consider a $d$-dimensional array $\{V_n, n \in \mathbb{Z}_+^d\}$ of independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space.
2 Mean convergence theorem

\( \mathcal{X} \) with norm \( \| \cdot \| \). In the current work, we establish the convergence in mean of order \( p (1 \leq p < 2) \) of the sums \( \sum_{j \leq n} V_j/|n|^{1/p} \), \( n \in \mathbb{Z}_d^+ \), under the condition that \( \{ \| V_n \|^p, n \in \mathbb{Z}_d^+ \} \) is uniformly integrable. The main results of this paper are Theorems 2.1 and 2.2. Theorem 2.1 is a stable-type \( p \) Banach space version of the main result of Thanh [11]. While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we will show in Theorem 2.2 that the implication in Theorem 2.1 indeed completely characterizes stable-type \( p \) Banach spaces.

Let \( 0 < p \leq 2 \) and let \( \{ \theta_n, n \geq 1 \} \) be independent and identically distributed stable random variables each with characteristic function \( \phi(t) = \exp\{-|t|^p \} \). The real separable Banach space \( \mathcal{X} \) is said to be of stable-type \( p \) if \( \sum_{n=1}^{\infty} \theta_n v_n \) converges a.s. whenever \( v_n \in \mathcal{X}, n \geq 1 \) with \( \sum_{n=1}^{\infty} \| v_n \|^p < \infty \). Equivalent characterizations of a Banach space being of stable-type \( p \), properties of stable-type \( p \) Banach spaces, as well as various relationships between the conditions Rademacher-type \( p \), and stable-type \( p \) may be found by Woyczyński in [13], by Marcus and Woyczyński in [7], and by Pisier in [8], see also the discussion by Adler et al. in [1]. We now mention explicitly some characterizations of this concept. The first theorem was obtained by Mandrekar and Zinn [6] and by Marcus and Woyczyński [7].

**Theorem 1.1.** Let \( 1 \leq p < 2 \) and let \( \mathcal{X} \) be a real separable Banach space. Then the following statements are equivalent.

(i) \( \mathcal{X} \) is of stable-type \( p \).

(ii) For every symmetric random elements \( V \), the condition \( n^p P(\| V \| > n) \to 0 \) as \( n \to \infty \) implies that

\[
\frac{\sum_{j=1}^{n} V_j}{n^{1/p}} \to 0 \quad \text{in probability},
\]

where \( \{ V_j, j \geq 1 \} \) are independent copies of \( V \).

**Theorem 1.2** (see [13, Theorem V.9.3]). Let \( 1 \leq p < 2 \) and let \( \mathcal{X} \) be a real separable Banach space. Then the following statements are equivalent.

(i) \( \mathcal{X} \) is of stable-type \( p \).

(ii) For each bounded sequence \( \{ x_n, n \geq 1 \} \) of elements of \( \mathcal{X} \),

\[
\frac{\sum_{j=1}^{n} x_k \epsilon_k}{n^{1/p}} \to 0 \quad \text{a.s.,}
\]

where \( \{ \epsilon_n, n \geq 1 \} \) is a Rademacher sequence.

The symbol \( C \) denotes throughout a generic constant \( (0 < C < \infty) \) which is not necessarily the same one in each appearance.

2 Main results

We can now present the main results. Theorem 2.1 is a stable-type \( p \) Banach space version of the main result of Thanh [11].
Theorem 2.1. Let \( \{V_n, n \in \mathbb{Z}_+^d\} \) be a \( d \)-dimensional array of independent mean-zero random elements in a real separable stable-type \( p \) \((1 \leq p < 2)\) Banach space \( \mathcal{X} \). If
\[
\{\|V_n\|^p, n \in \mathbb{Z}_+^d\} \text{ is uniformly integrable,} \tag{2.1}
\]
then
\[
\frac{\sum_{j \prec n} V_j}{|n|^{1/p}} \rightarrow 0 \text{ in } L^p \text{ as } |n| \rightarrow \infty. \tag{2.2}
\]

Proof. For arbitrary \( \epsilon > 0 \), there exists \( M > 0 \) such that
\[
E(\|V_n\|^p I(\|V_n\| > M)) < \epsilon, \quad \forall n \in \mathbb{Z}_+^d. \tag{2.3}
\]
Set
\[
V_n' = V_n I(\|V_n\| \leq M), \quad n \in \mathbb{Z}_+^d, \quad V_n'' = V_n I(\|V_n\| > M), \quad n \in \mathbb{Z}_+^d. \tag{2.4}
\]
Since \( \mathcal{X} \) is of stable-type \( p \) and \( p < 2 \), it is of Rademacher-type \( q \) for some \( p < q < 2 \). Thus
\[
E \left( \sum_{j \prec n} |V_j|^p \right) \leq 2^{p-1} \left( E \left( \sum_{j \prec n} (V_j' - EV_j')^p \right) + C \sum_{j \prec n} \|V_j'' - EV_j''\|^p \right) \leq 2^{p-1} \left( E \left( \sum_{j \prec n} (V_j' - EV_j')^q \right)^{p/q} + C \sum_{j \prec n} \|V_j'' - EV_j''\|^p \right) \leq 2^{p-1} \left( E \left( \sum_{j \prec n} (V_j' - EV_j')^q \right)^{p/q} \right)^{p/q} + C \sum_{j \prec n} \|V_j'' - EV_j''\|^p \leq C \left( \sum_{j \prec n} E \|V_j' - EV_j'\|^q \right)^{p/q} + C \sum_{j \prec n} \|V_j''\|^p \leq C (|n|M^q)^{p/q} + C |n| \epsilon = o(|n|), \quad \text{as } |n| \rightarrow \infty. \tag{2.5}
\]

While the proof of Theorem 2.1 and the proof of the main result in Thanh [11] are similar, we now show in Theorem 2.2 that the implication ((2.1) \( \Rightarrow \) (2.2)) in Theorem 2.1 indeed completely characterizes stable-type \( p \) Banach spaces.
4 Mean convergence theorem

Theorem 2.2. Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Then the following statements are equivalent.

(i) $\mathcal{X}$ is of stable-type $p$.

(ii) For every $d$-dimensional array $\{V_n, n \in \mathbb{Z}_d^d\}$ of independent mean-zero random elements in $\mathcal{X}$, the condition (2.1) implies (2.2).

(iii) For every $d$-dimensional array $\{V, V_n, n \in \mathbb{Z}_d^d\}$ of independent mean-zero random elements in $\mathcal{X}$, the conditions

$$E\|V\|^p < \infty, \quad \sup_{n \in \mathbb{Z}_d^d} P\{\|V_n\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0,$$

imply (2.2).

Proof. The implication ((i) $\Rightarrow$ (ii)) is precisely Theorem 2.1, whereas the implication ((ii) $\Rightarrow$ (iii)) is immediate. It remains to verify the implication ((iii) $\Rightarrow$ (i)). For reasons of clarity, we collect some of the steps in the following lemmas. The first lemma is a slight modification of de Acosta [4, Theorem 3.1] which holds for sequences of independent identically distributed random elements. The proof of the following modification can be obtained from de Acosta [4, Theorem 3.1] line by line, and so will be omitted. □

Lemma 2.3. Let $\mathcal{X}$ be a real separable Banach space, $1 \leq p < 2$. Let $\{V, W_k, k \geq 1\}$ be sequence of independent random elements such that $E\|V\|^p < \infty$ and $\sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}$ for all $t > 0$. Then

$$\lim_{n \to \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{in probability}$$

(2.7)

if and only if

$$\lim_{n \to \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{a.s.}$$

(2.8)

Lemma 2.4. Let $1 \leq p < 2$ and let $\mathcal{X}$ be a real separable Banach space. Suppose that for every sequence $\{V, W_k, k \geq 1\}$ of independent mean-zero random elements in $\mathcal{X}$, the conditions

$$E\|V\|^p < \infty, \quad \sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0,$$

imply that

$$\lim_{n \to \infty} \frac{\sum_{k=1}^n W_k}{n^{1/p}} = 0 \quad \text{in probability.}$$

(2.10)

Then $\mathcal{X}$ is of stable-type $p$.

Proof of Lemma 2.4. Let $\{\varepsilon_k, k \geq 1\}$ be a Rademacher sequence and let $\{x_k, k \geq 1\}$ be a sequence of elements in $\mathcal{X}$ such that

$$\sup_{k \geq 1} \|x_k\| < \infty.$$
Then by the hypothesis of the lemma,
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} k x_k}{n^{1/p}} = 0 \quad \text{in probability.} \tag{2.12}
\]
By Lemma 2.3,
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} k x_k}{n^{1/p}} = 0 \quad \text{a.s.} \tag{2.13}
\]
Hence, by Theorem 1.2, \(X\) is of stable-type \(p\). The proof of Lemma 2.4 is completed. \(\square\)

We now prove the implication ((iii) \(\Rightarrow\) (i)). If \(d = 1\), then the conclusion follows directly from Lemma 2.4. So, we can assume that \(d \geq 2\). Let \(\{V, W_k, k \geq 1\}\) be a sequence of independent mean-zero random elements in \(X\) such that \(E\|V\|^p < \infty\) and \(\sup_{k \geq 1} P\{\|W_k\| > t\} \leq CP\{\|V\| > t\}\) for all \(t > 0\). For \(n = (n_1, \ldots, n_d) \in \mathbb{Z}^d_+\), set
\[
V_{(n_1, \ldots, n_d)} = W_{n_1}, \quad \text{if } n_2 = \cdots = n_d = 1,
\]
\[
V_{(n_1, \ldots, n_d)} = 0, \quad \text{if } \max\{n_2, \ldots, n_d\} \geq 2. \tag{2.14}
\]
Then \(\{V_n, n \in \mathbb{Z}^d\}\) is an array of independent mean-zero random elements, and
\[
\sup_{n \in \mathbb{Z}^d} P\{\|V_n\| > t\} \leq CP\{\|V\| > t\}, \quad \forall t > 0. \tag{2.15}
\]
By (iii),
\[
\frac{1}{|n|^{1/p}} \sum_{j < n} V_j \longrightarrow 0 \quad \text{in } L^p \text{ as } |n| \longrightarrow \infty. \tag{2.16}
\]
This implies by taking \(n_2 = \cdots = n_d = 1\) and letting \(n_1 \to \infty\) that
\[
\frac{1}{n_1^{1/p}} \sum_{k=1}^{n_1} W_k \longrightarrow 0 \quad \text{in } L^p, \text{ so in probability as } n_1 \longrightarrow \infty. \tag{2.17}
\]
By Lemma 2.4, \(X\) is of stable-type \(p\).

Remark 2.5. In Theorem 2.1, if \(0 < p < 1\), then the independence hypothesis and the hypothesis that the \(\{V_n, n \in \mathbb{Z}^d\}\) have mean-zero are not needed for the theorem to hold.

Indeed, for arbitrary \(\epsilon > 0\), define \(V'_n\) and \(V''_n, n \in \mathbb{Z}^d\) as in the proof of Theorem 2.1. If \(0 < p < 1\), then
\[
E \left( \sum_{j < n} V_j \right)^p \leq E \left( \sum_{j < n} V'_j \right)^p + \sum_{j < n} E \left( V''_j \right)^p \leq \left( |n| M \right)^p + |n| \epsilon = o(|n|), \quad \text{as } |n| \longrightarrow \infty. \tag{2.18}
\]
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References


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