A POSITIVE SOLUTION FOR SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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This paper presents new existence results for the singular discrete boundary value problem

\[-\Delta^2 u(k-1) = g(k,u(k)) + \lambda h(k,u(k)), \quad k \in [1,T], \quad u(0) = 0 = u(T+1)\]

In particular, our nonlinearity may be singular in its dependent variable and is allowed to change sign.

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1. Introduction

Let \(a, b\) \((b > a)\) be nonnegative integers. We define the discrete interval \([a, b] = \{a, a+1, \ldots, b\}\). All other intervals will carry its standard meaning, for example, \([0, \infty)\) denotes the set of nonnegative real numbers. The symbol \(\Delta\) denotes the forward difference operator with step size 1, that is, \(\Delta u(k) = u(k+1) - u(k)\). Furthermore for a positive \(m\), \(\Delta^m\) is defined as \(\Delta^m u(k) = \Delta^{m-1}(\Delta u(k))\). In this paper, we will study positive solutions of the second-order discrete boundary value problem

\[-\Delta^2 u(k-1) = g(k,u(k)) + \lambda h(k,u(k)), \quad k \in [1,T], \quad u(0) = 0 = u(T+1), \quad (1.1)\]

where \(\lambda > 0\) is a constant and \(T > 2\) is a positive integer. Here, \(g : [1, T] \times (0, \infty) \rightarrow \mathbb{R}\) and \(h : [1, T] \times [0, \infty) \rightarrow [0, \infty)\) are continuous. As a result, our nonlinearity may be singular at \(u = 0\) and may change sign.

By a solution \(u\) of the boundary value problem (1.1), we mean \(u : [0, T+1] \rightarrow \mathbb{R}\), \(u\) satisfies the difference equation (1.1) on \([1, T]\) and the stated boundary data.

We will let \(C[0, T+1]\) denote the class of map \(u\) continuous on \([0, T+1]\) (discrete topology), with norm \(|u|_{\infty} = \max_{k \in [0, T+1]} |u(k)|\).

2. Main results

The main result of the paper is the following.
Theorem 2.1. Suppose the following conditions hold:

(G) there exist \( g_i : [1, T] \times (0, \infty) \rightarrow (0, \infty) \) \((i = 1, 2)\) continuous functions such that
\[
\begin{align*}
    g_i(k, \cdot) & \text{ is strictly decreasing for } k \in [1, T], \\
    -g_1(k, u) & \leq g(k, u) \leq g_2(k, u) \text{ for } (k, u) \in [1, T] \times (0, \infty), \\
    \int_0^1 g_1(k, s)ds & < \infty \quad \text{for } k \in [1, T], \\
    \forall s_0 > 0, \sup_{s \leq s_0} \left| \frac{\partial}{\partial s} g_2(\cdot, s) \right| & \in C[1, T];
\end{align*}
\]

(H) there exist \( h_i : [1, T] \times [0, \infty) \rightarrow (0, \infty) \) \((i = 1, 2)\) continuous functions such that
\[
\begin{align*}
    h_i(k, \cdot) & \text{ increasing for } k \in [1, T], \\
    h_1(k, u) & \leq h(k, u) \leq h_2(k, u) \text{ for } (k, u) \in [1, T] \times (0, \infty), \\
    \lim_{u \to \infty} \frac{h_2(k, u)}{u} & = 0 \quad \text{for } k \in [1, T], \\
    \text{there exists } \bar{s} > 0 \text{ such that } h_1(k, \bar{s}) > 0 \text{ for all } k \in [1, T].
\end{align*}
\]

Then there exists \( \lambda_0 \geq 0 \) such that for every \( \lambda \geq \lambda_0 \), problem (1.1) has at least one solution \( u \in C[0, T + 1] \text{ and } u(k) > 0 \text{ for } k \in [1, T]. \) Moreover, there exists \( c_i = c_i(\lambda, g, h, \phi_1) > 0 \) \((i = 1, 2)\) such that
\[
\begin{align*}
    c_1 \phi_1(k) & \leq u(k) \leq c_2(\phi_1(k) + 1) \quad \text{for } k \in [0, T + 1],
\end{align*}
\]
where \( \phi_1 \) is defined in Lemma 2.2.

It is worth remarking here that an estimate for \( \lambda_0 \) will be given in the proof of Lemma 2.11.

We first give some lemmas which will help us to prove Theorem 2.1.

Lemma 2.2 [1]. Consider the following eigenvalue problem:
\[
\begin{align*}
    -\Delta^2 u(k - 1) & = \lambda u(k), \quad k \in [1, T], \\
    u(0) & = u(T + 1) = 0.
\end{align*}
\]

Then the eigenvalues are
\[
\lambda_m = 4 \sin^2 \frac{m\pi}{2(T + 1)}, \quad 1 \leq m \leq T,
\]
and the corresponding eigenfunctions are
\[
\phi_m(k) = \sin \frac{mk\pi}{T + 1} \quad \text{for } k \in [0, T + 1], 1 \leq m \leq T.
\]

Lemma 2.3 [3]. Let \( G_a(k, l) \) be Green's function of the BVP
\[
\begin{align*}
    -\Delta^2 u(k - 1) + a(t)u(t) & = 0 \quad \text{for } t \in [1, T], \\
    u(0) & = 0, \quad u(T + 1) = 0.
\end{align*}
\]
Lemma 2.5. The following statements hold:

(i) for any $f \in C[1, T]$, (2.10) is uniquely solvable and $u = A(f)$;

(ii) if $f(k) \geq 0$ for $k \in [1, T]$, then the solution of (2.10) is nonnegative.

Corollary 2.6. If $f_1(k) \leq f_2(k)$ for $k \in [1, T]$, then $A(f_1)(k) \leq A(f_2)(k)$ for $k \in [1, T]$.

Lemma 2.7. Suppose (G) and (H) hold. Let $n_0 \in \mathbb{N}$. Assume that for every $n > n_0$, there exist $a_n \in C[1, T]$, $0 \leq a_n$, and there exist $\bar{u}, \bar{u}_n, \hat{u}_n, \tilde{u} \in C[0, T + 1]$ such that

$$0 < \bar{u}(k) \leq \bar{u}_n(k) \leq \hat{u}_n(k) \leq \tilde{u}(k) \quad \text{for } k \in [1, T],$$

(2.13)

and $\hat{u}(0) = \hat{u}(T + 1) = 0$. If

$$-\Delta^2 \bar{u}_n(k - 1) + a_n(k)\bar{u}_n(k)$$

$$\leq g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k)v(k) \quad \text{for } k \in [1, T],$$

(2.14)

and

$$-\Delta^2 \hat{u}_n(k - 1) + a_n(k)\hat{u}_n(k)$$

$$\geq g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k)v(k) \quad \text{for } k \in [1, T],$$

(2.15)
where $\lambda \geq 0$ and $v \in [\bar{u}_n, \hat{u}_n] = \{ u \in C[0, T+1], \bar{u}_n(k) \leq u(k) \leq \hat{u}_n(k) \text{ for } k \in [0, T+1]\}$, then problem (1.1) has a solution $u \in C[0, T+1]$ such that $\bar{u}(k) \leq u(k) \leq \hat{u}(k)$ for $k \in [0, T+1]$.

**Proof.** Fix $v \in [\bar{u}, \hat{u}]$. From Lemma 2.5, there exists $\Psi(v) \in C[0, T+1]$ such that

$$-\Delta^2 \Psi(v)(k-1) + a_n(k) \Psi(v)(k) = g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k) v(k) \quad \text{for } k \in [1, T],$$

$$\Psi(v)(0) = \Psi(v)(T+1) = 0.$$  \hfill (2.16)

Then

$$\Psi(v)(k) = A \left( g\left(\cdot, \frac{1}{n} + v\right) + \lambda h(\cdot, v) + a_n v\right)(k) \quad \text{for } k \in [1, T].$$  \hfill (2.17)

Note also that $\Psi : C[0, T+1] \to C[0, T+1]$ is a completely continuous operator. By Corollary 2.6, we have

$$\bar{u}_n(k) \leq \Psi(v)(k) \leq \hat{u}_n(k) \quad \text{for } k \in [0, T+1].$$  \hfill (2.18)

From Schauder’s fixed point theorem (note that $\Psi z : [\bar{u}, \hat{u}] \to [\bar{u}, \hat{u}]$), there exists $u_n \in C[0, T+1]$ such that $\bar{u}_n(k) \leq u_n(k) \leq \hat{u}_n(k)$ and $\Psi(u_n)(k) = u_n(k)$ for $k \in [1, T]$. Note that

$$-\Delta^2 u_n(k-1) = g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \quad \text{for } k \in [1, T],$$

$$u_n(0) = u_n(T+1) = 0.$$  \hfill (2.19)

Let $m := \min\{\bar{u}(k) : k \in [1, T]\} > 0$ and $M := \max\{\hat{u}(k) : k \in [1, T]\}$. Then

$$m \leq u_n(k) \leq M \quad \text{for } k \in [1, T], n = 1, 2, \ldots,$$  \hfill (2.20)

and for $k \in [1, T]$, we have

$$\left| g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \right| \leq g_2(k, m) + \lambda h_2(k, M).$$  \hfill (2.21)

From the Arzela-Ascoli theorem, there exist a $u \in C[0, T+1]$ and a subsequence $\{u_{n_m}\}_{m \in \mathbb{N}}$ converging to $u$ in $C[0, T+1]$, and of course

$$u(k) = \lim_{m \to \infty} u_{n_m}(k) \quad \text{for } k \in [0, T+1].$$  \hfill (2.22)
Observe that \( u_{n_0} \in [\bar{u}, \hat{u}] \), so \( u(0) = u(T + 1) = 0 \) and \( u \in C[0, T + 1] \) with \( u > 0 \) in \([1, T]\).

Also,

\[
    u(k) = \lim_{m \to \infty} \sum_{l=1}^{T} G_0(k, l) \left[ g\left( l, \frac{1}{n} + u_{n_0}(l) \right) + \lambda h(l, u_{n_0}(l)) \right]
    = \sum_{l=1}^{T} G_0(k, l) \left[ g(l, u(l)) + \lambda h(l, u(l)) \right].
\]

(2.23)

As a result

\[
    -\Delta^2 u(k - 1) = g(k, u(k)) + \lambda h(k, u(k)) \quad \text{for } k \in [1, T],
    \quad u(0) = u(T + 1) = 0.
\]

(2.24)

**Lemma 2.8.** Let \( \psi : [1, T] \times (0, \infty) \to (0, \infty) \) be a continuous function with \( \psi(k, \cdot) \) strictly decreasing. Then the problem

\[
    -\Delta^2 \omega(k - 1) = \psi\left( k, \omega + \frac{1}{n} \right) \quad \text{for } k \in [0, T],
    \quad \omega(0) = \omega(T + 1) = 0
\]

has a solution \( \omega_n \in C[0, T + 1] \) such that

\[
    \omega_n(k) \leq \omega_{n+1}(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [0, T + 1], \ n \in \mathbb{N}.
\]

(2.26)

If \( \omega(k) = \lim_{n \to \infty} \omega_n(k) \) for \( k \in [0, T + 1] \), then

\[
    \omega \in C[0, T + 1], \quad \omega(k) > 0, \quad \text{for } k \in [1, T],
    \quad -\Delta^2 \omega(k - 1) = \psi(k, \omega) \quad \text{for } k \in [1, T],
    \quad \omega(0) = \omega(T + 1) = 0.
\]

(2.27)

**Proof.** There exists \( \chi_1 \in C[0, T + 1] \) such that

\[
    -\Delta^2 \chi_1(k - 1) = \psi(k, 1),
    \quad \chi_1(0) = \chi_1(T + 1) = 0,
    \quad \chi_1(k) > 0 \quad \text{for } k \in [1, T].
\]

(2.28)

Notice that

\[
    -\Delta^2 \chi_1(k - 1) = \psi(k, 1) \geq \psi(k, 1 + \chi_1(k)),
    \quad 0 \leq \psi(k, 1 + 0).
\]

(2.29)

By a standard upper-lower solution method [2, page 264], there exists \( \omega_1 \in C[0, T + 1] \) such that

\[
    -\Delta^2 \omega_1(k - 1) = \psi(k, 1 + \omega_1(k)) \quad \text{for } k \in [1, T],
    \quad \omega_1(0) = \omega_1(T + 1) = 0.
\]

(2.30)
Suppose that there exists $\omega_n \in C[0, T + 1]$ such that

$$-\Delta^2 \omega_n(k - 1) = \psi\left(k, \frac{1}{n} + \omega_n(k)\right),$$

$$\omega_n(0) = \omega_n(T + 1) = 0,$$  \hspace{1cm} (2.31)

$$\omega_n(k) > 0 \text{ for } k \in [1, T].$$

We know that there exist $\chi_{n+1} \in C[0, T + 1]$ such that

$$-\Delta^2 \chi_{n+1}(k - 1) = \psi\left(k, \frac{1}{n+1}\right),$$

$$\chi_{n+1}(0) = \chi_{n+1}(T + 1) = 0,$$  \hspace{1cm} (2.32)

$$\chi_{n+1}(k) > 0 \text{ for } k \in [1, T].$$

Then

$$-\Delta^2 \chi_{n+1}(k - 1) = \psi\left(k, \frac{1}{n+1}\right) \geq \psi\left(k, \frac{1}{n+1} + \omega_n(k)\right),$$

$$-\Delta^2 \omega_n(k - 1) = \psi\left(k, \frac{1}{n+1} + \omega_n(k)\right) \leq \psi\left(k, \frac{1}{n+1} + \omega_n(k)\right) \text{ for } k \in [1, T],$$

$$\omega_n(0) = \omega_n(T + 1) = 0,$$

$$\omega_n(k) = \sum_{l=1}^{T} G_0(k, l)\psi\left(l, \frac{1}{n} + \omega_n(l)\right) \leq \sum_{l=1}^{T} G_0(k, l)\psi\left(l, \frac{1}{n+1}\right) = \chi_{n+1}(k) \text{ for } k \in [1, T].$$ \hspace{1cm} (2.33)

By a standard upper-lower solution method, there exist $\omega_{n+1} \in C[0, T + 1]$ such that

$$-\Delta^2 \omega_{n+1}(k - 1) = \psi\left(k, \frac{1}{n+1} + \omega_{n+1}\right) \text{ for } k \in [1, T],$$

$$\omega_{n+1}(0) = \omega_{n+1}(T + 1) = 0,$$  \hspace{1cm} (2.34)

$$\omega_n(k) \leq \omega_{n+1}(k) \text{ for } k \in [0, T + 1].$$

Next we prove

$$\omega_{n+1}(k) + \frac{1}{n+1} \leq \omega_n(k) + \frac{1}{n} \text{ for } k \in [0, T + 1].$$ \hspace{1cm} (2.35)

To see this, we consider the problem

$$-\Delta^2 v(k - 1) = \psi(k, v) \text{ for } k \in [1, T],$$

$$v(0) = v(T + 1) = \frac{1}{n}.$$ \hspace{1cm} (2.36)

Then $v_n(k) = 1/n + \omega_n(k)$ for $k \in [0, T + 1]$ is a solution of (2.36)_n. We next prove

$$v_{n+1}(k) \leq v_n(k) \text{ for } k \in [0, T + 1].$$ \hspace{1cm} (2.37)
Since \( v_{n+1}(0) = 1/(n+1) < 1/n = v_n(0) \), \( v_{n+1}(1) = 1/(n+1) < 1/n = v_n(1) \), we need only to prove that

\[
v_{n+1}(k) \leq v_n(k) \quad \text{for } k \in [1, T]. \tag{2.38}
\]

If this is not true, then there exist \( m \in [1, T] \) with \( v_{n+1}(m) > v_n(m) > 0 \). Let \( \sigma \) be the point where \( v_{n+1}(k) - v_n(k) \) assumes its maximum over \([1, T]\). Certainly, \( v_{n+1}(\sigma) - v_n(\sigma) > 0 \). Let \( y(k) = v_{n+1}(k) - v_n(k) \). Now \( y(\sigma) \geq y(\sigma + 1) \) and \( y(\sigma) \geq y(\sigma - 1) \) imply that

\[
2y(\sigma) \geq y(\sigma + 1) + y(\sigma - 1), \tag{2.39}
\]

that is,

\[
y(\sigma + 1) + y(\sigma - 1) - 2y(\sigma) \leq 0. \tag{2.40}
\]

Thus

\[
\Delta^2 y(\sigma - 1) \leq 0. \tag{2.41}
\]

On the other hand, since \( v_{n+1}(\sigma) > v_n(\sigma) \), we have

\[
\Delta^2 y(\sigma - 1) = \Delta^2 v_{n+1}(\sigma - 1) - \Delta^2 v_n(\sigma - 1) = -\psi(\sigma, v_{n+1}(\sigma)) + \psi(\sigma, v_n(\sigma)) = \psi(\sigma, v_n(\sigma)) - \psi(\sigma, v_{n+1}(\sigma)) > 0. \tag{2.42}
\]

This is a contradiction. Thus \( v_{n+1}(k) \leq v_n(k) \) for \( k \in [1, T] \), and so

\[
0 < \frac{1}{n+1} + \omega_{n+1} \leq \omega_n + \frac{1}{n}. \tag{2.43}
\]

Also notice that

\[
\omega_1(k) \leq \omega_n(k) \leq \omega_{n+1}(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [0, T + 1], n \in \mathbb{N}. \tag{2.44}
\]

Now with

\[
\omega(k) = \lim_{n \to \infty} \omega_n(k) = \sup_{n \in \mathbb{N}} \omega_n(k) \quad \text{for } k \in [0, T + 1], \tag{2.45}
\]

we have

\[
0 < \omega_1(k) \leq \omega(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [1, T],
\]

\[
\omega(0) = \omega(T + 1) = 0. \tag{2.46}
\]
Also for \( k \in [1, T] \), we have
\[
\omega(k) = \lim_{n \to \infty} \omega_n(k)
\]
\[
= \lim_{n \to \infty} \sum_{l=1}^{T} G(k,l) \psi(l, \frac{1}{n} + \omega_n(l))
\]
\[
= \sum_{l=1}^{T} G(k,l) \psi(l, \omega(l)),
\]
so
\[
-\Delta^2 \omega(k - 1) = \psi(k, \omega) \quad \text{for } k \in [0, T],
\]
\[
\omega(0) = \omega(T + 1) = 0.
\] (2.48)

**Lemma 2.9.** Suppose that \( m : [1, T] \times [0, \infty) \to [0, \infty) \) is a continuous function such that
\[
m(k, \cdot) \text{ is increasing,}
\]
\[
\lim_{u \to +\infty} \frac{m(k,u)}{u} = 0 \quad \text{for } k \in [1, T].
\] (2.49)

There exist \( R_0 > 0 \) and \( \tilde{v} \in C[0, T + 1] \) with \( 0 \leq \tilde{v} \leq R_0 \phi_1 \) and
\[
-\Delta^2 \tilde{v}(k - 1) = m(k, \tilde{v}) \quad \text{for } k \in [1, T],
\]
\[
\tilde{v}(0) = \tilde{v}(T + 1) = 0.
\] (2.50)

**Proof.** We first prove that
\[
\lim_{R \to \infty} \sum_{l=1}^{T} G_0(k,l)m(l, v(l)) \leq R \phi_1(k) = 0 \quad \text{for } k \in [1, T],
\] (2.51)
for all \( v \in C[0, T + 1] \) with \( 0 \leq v(i) \leq R \phi_1(i) \) for \( i \in [0, T + 1] \).

From (2.49), for all \( \sigma > 0 \), there exist \( s_\sigma > 0 \) such that
\[
m(k, s) \leq \sigma s \quad \text{for } k \in [1, T] \text{ and } s_\sigma \leq s.
\] (2.52)

As a result,
\[
m(k, v(k)) \big|_{0 \leq v(k) \leq R \phi_1(k)} \leq m(k, s_\sigma) + \sigma v(k) \leq m(k, s_\sigma) + \sigma R \phi_1(k) \quad \text{for } k \in [1, T],
\] (2.53)
so
\[
\frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l)m(l, v(l)) \leq \frac{1}{\phi_1(k)} \left[ \sum_{l=1}^{T} G_0(k,l)m(l, s_\sigma) + R \sigma \sum_{l=1}^{T} G_0(k,l) \phi_1(l) \right]
\]
\[
= \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l)m(l, s_\sigma) + \frac{R \sigma}{A_1},
\] (2.54)
and consequently
\[
\frac{1}{R \Phi_1(k)} \sum_{l=1}^{T} G_0(k,l)m(l,v(l)) \leq \frac{1}{R \Phi_1(k)} \sum_{l=1}^{T} G_0(k,l)m(l,s_\sigma) + \frac{\sigma}{\lambda_1},
\]
so (2.51) follows. Thus there exist \( R_0 > 0 \) such that if \( v \in C[0, T + 1] \) and \( 0 \leq v(i) \leq R_0 \Phi_1(i) \) for \( i \in [0, T + 1] \), then
\[
\frac{1}{R_0 \Phi_1(k)} \sum_{l=1}^{T} G_0(k,l)m(l,v(l)) \leq 1 \quad \text{for } k \in [1, T],
\]
and so
\[
0 \leq \sum_{l=1}^{T} G_0(k,l)m(l,v(l)) \leq R_0 \Phi_1(k) \quad \text{for } k \in [1, T].
\]

Let \( \Phi : C[1, T] \to C[1, T] \) be the operator defined by
\[
(\Phi v)(k) := \sum_{l=1}^{T} G_0(k,l)m(l,v(l)) \quad \text{for } v \in C[1, T], \ k \in [1, T].
\]

It is easy to see that \( \Phi \) is a completely continuous operator. Also if \( v \in C[0, T + 1] \) and \( 0 \leq v(k) \leq R_0 \Phi_1(k) \) for \( k \in [1, T] \), then \( 0 \leq \Phi(v)(k) \leq R_0 \Phi_1(k) \) for \( k \in [1, T] \), so Schauder’s fixed point theorem guarantees that there exists \( \tilde{v} \in [0, R_0 \Phi_1] \) with \( \Phi(\tilde{v}) = \tilde{v} \), that is,
\[
-\Delta^2 \tilde{v}(k - 1) = m(k, \tilde{v}), \quad \tilde{v}(0) = \tilde{v}(T + 1) = 0.
\]

**Corollary 2.10.** Let \( \psi(k,s), m(k,s), (\omega_n)_{n \in \mathbb{N}}, \) and \( R_0 > 0 \) be as in Lemmas 2.8 and 2.9. Then there exist \( \{\tilde{v}_n\}_{n \in \mathbb{N}} \subset C[0, T + 1] \) and \( 0 \leq \tilde{v}_n \leq R_0 \Phi_1 \) such that
\[
-\Delta^2 \tilde{v}_n(k - 1) = m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T],
\]
\[
\tilde{v}_n(0) = \tilde{v}_n(T + 1) = 0,
\]
\[
-\Delta^2 (w_n + \tilde{v}_n)(k - 1) \geq \psi \left(k, \frac{1}{n} + \omega_n + \tilde{v}_n\right) + m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T].
\]

**Proof.** Let \( n \in \mathbb{N} \) be fixed. Then \( m(k, \omega_n + s) \) satisfies the conditions of Lemma 2.9, so there exists \( \tilde{v}_n \in C[0, T + 1] \) with \( 0 \leq \tilde{v}_n \leq R_0 \Phi_1 \) such that (2.60) holds and
\[
-\Delta^2 (w_n + \tilde{v}_n)(k - 1) = -\Delta^2 w_n(k - 1) - \Delta^2 \tilde{v}_n(k - 1) = \psi \left(k, \frac{1}{n} + \omega_n\right) + m(k, \omega_n + \tilde{v}_n)
\]
\[
\geq \psi \left(k, \frac{1}{n} + \omega_n + \tilde{v}_n\right) + m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T].
\]
\[
\Box
\]
Lemma 2.11. Suppose (G) and (H) hold. Then there exist \( \lambda_0 \geq 0, c > 0 \) such that for all \( \lambda \geq \lambda_0 \), there exist \( R_c > c, \overline{u} \in C([0, T + 1]) \) with \( c \phi_1(k) \leq \overline{u}(k) \leq R_c \phi_1(k) \) and

\[
-\Delta^2 \overline{u}(k - 1) = -g_1(k, \overline{u}) + \lambda h_1(k, \overline{u}) \quad \text{for } k \in [1, T],
\]

\[
\overline{u}(0) = \overline{u}(T + 1) = 0.
\] (2.62)

Proof. Let us consider the operator \( T_\lambda : C[1, T] \to C[1, T] \) given by

\[
T_\lambda(v)(k) := \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k, l) \left[ -g_1(l, v(l)\phi_1(l)) + \lambda h_1(l, v(l)\phi_1(l)) \right] \quad \text{for } k \in [1, T].
\] (2.63)

By (H), there exists \( s \geq 0 \) such that \( 0 < h_1(k, s) \) for \( k \in [1, T] \). We let

\[
c = 2 \frac{s + 1}{|\phi_1|_{\infty}}, \quad \Theta = \left\{ k \in [1, T] : \frac{|\phi_1|_{\infty}}{2} < \phi_1(k) \right\}.
\] (2.64)

Note that \( \Theta \) is nonempty. If \( k \in \Theta, v \in C[0, T + 1] \), and \( c \leq v \), we have

\[
\overline{s} = \frac{c |\phi_1|_{\infty}}{2} - 1 \leq \frac{c |\phi_1|_{\infty}}{2} \leq c \phi_1(k) \leq v(k)\phi_1(k),
\] (2.65)

so

\[
h_1(k, \overline{s}) \leq h_1(k, v(k)\phi_1(k)),
\] (2.66)

for all \( v \in C[0, T + 1] \) with \( c \leq v \). Let

\[
\rho = \min_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \overline{s}) > 0,
\] (2.67)

and note for \( v \in C[0, T + 1] \) with \( c \leq v \) that

\[
\frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k, l) h_1(l, v(l)\phi_1(l)) \geq \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, v(l)\phi_1(l)) \\
\geq \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \overline{s}) \quad (\text{see (2.66)}) \\
\geq \min_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \overline{s}) \\
= \rho \quad \forall k \in [1, T],
\] (2.68)

that is,

\[
\frac{\phi_1(k)}{\sum_{l=1}^{T} G_0(k, l) h_1(l, v(l)\phi_1(l))} \leq \frac{1}{\rho}.
\] (2.69)
On the other hand, for all $v \in C[0, T + 1]$ with $v \geq c$, we have

$$c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l))$$

$$\leq c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \phi_1(l)) \leq c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \mu),$$

(2.70)

where $\mu = \min_{1 \leq l \leq T} \phi_1(l)$. Thus, for all $v \in C[0, T + 1]$ with $v(k) \geq c$, we have

$$c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) \leq \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \mu) \leq \frac{1}{\rho} \left( c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \mu) \right)$$

for $k \in [1, T]$.

Let

$$\lambda_0 := \sup \left\{ \left| c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) \right| : v \in C[0, T + 1], \ c \leq v \right\} < \infty,$$

(2.72)

where $|u|_* = \max [1, T] |u(k)|$. Then, for all $\lambda \geq \lambda_0$, $v \in C[0, T + 1]$, and $c \leq v$, we have for $k \in [1, T]$ that

$$c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) \leq \lambda,$$

(2.73)

that is,

$$c + \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) \leq \frac{\lambda}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) h_1(l, v(l) \phi_1(l)),$$

(2.74)

so

$$c \leq \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) \left( - g_1(l, v(l) \phi_1(l)) + \lambda h_1(l, v(l) \phi_1(l)) \right)$$

$$= T_\lambda(v)(k) \quad \text{for } k \in [1, T].$$

(2.75)

On the other hand, for all $v \in C[0, T + 1]$ with $v \geq c$, we have

$$\frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) \leq \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \phi_1(l))$$

$$\leq \max_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) g_1(l, c \phi_1(l)),$$

(2.76)
Also let

\[
\phi_1(k) \sum_{l=1}^{T} G_0(k,l) g_1(l, v(l) \phi_1(l)) = 0,
\]

\[\text{(2.77)}\]

for all \( v \in C[0, T + 1] \) with \( v \geq c \) and \( k \in [1, T] \). Essentially the same reasoning as in the proof of (2.51) yields (note that \( \lim_{n \to \infty} (h_1(k, u)/u) = 0 \) for \( k \in [1, T] \))

\[
\lim_{R \to -\infty} \frac{1}{R} \left[ \frac{1}{\phi_1(k)} \sum_{l=1}^{T} G_0(k,l) h_1(l, v(l) \phi_1(l)) \right] = 0
\]

\[\text{(2.78)}\]

for all \( v \in C[0, T + 1] \) with \( 0 \leq v(i) \leq R \) and \( i \in [1, T] \). Thus if \( \lambda \geq \lambda_0 \), there exists \( R_c > c \) with \( T_{\lambda}([c, R_c]) \subset [c, R_c] \).

It is easy to see that \( T_{\lambda} \) is a completely continuous operator, so Schauder’s fixed point theorem guarantees that there exists \( \bar{v} \in [c, R_c] \) with \( T_{\lambda}(\bar{v}) = \bar{v} \), that is,

\[
\bar{v}(k) \phi_1(k) = \sum_{l=1}^{T} G_0(k,l) \left( -g_1(l, \bar{v}(l) \phi_1(l)) + \lambda h_1(l, \bar{v}(l) \phi_1(l)) \right).
\]

\[\text{(2.79)}\]

The function \( \bar{v} = \phi_1 \bar{v} \) satisfies (2.62). \( \square \)

**Proof of Theorem 2.1.** Let \( \lambda_0 > 0, c > 0 \), and \( \bar{v} \in (C[0, T + 1]) \) be defined as in Lemma 2.11. Also let

\[
\psi(k, s) = g_2(k, s) + \lambda h_1(k, \bar{v}(k)) \quad \text{for} \quad k \in [1, T],
\]

\[
m(k, s) = \lambda h_2(k, s),
\]

\[\text{(2.80)}\]

where \( \lambda \geq \lambda_0 \).

From (G), we notice that \( \psi \) satisfies the assumptions of Lemma 2.8. As a result, there exist \( \omega, \omega_n \in C[0, T + 1] \) such that

\[
-\Delta^2 \omega_n(k-1) = g_2 \left( k, \frac{1}{n} + \omega_n \right) + \lambda h_1(k, \bar{v}(k)) \quad \text{for} \quad k \in [1, T],
\]

\[
\omega_n(0) = \omega_n(T + 1) = 0,
\]

\[\text{(2.81)}\]

\[
\omega(k) = \lim_{n \to \infty} \omega_n(k) \quad \text{for} \quad k \in [0, T + 1].
\]

From (H), we notice that \( m \) satisfies the assumptions of Lemma 2.9. As a result from Corollary 2.10, there exist \( R_0 > 0 \) and \( \tilde{v}_n \in C([0, T + 1]) \), \( 0 \leq \tilde{v}_n(k) \leq R_0 \phi_1(k) \) for \( k \in [0, T + 1] \) such that

\[
-\Delta^2 \tilde{v}_n(k-1) = \lambda h_2(k, \omega_n + \tilde{v}_n) \quad \text{for} \quad k \in [1, T],
\]

\[
\tilde{v}_n(0) = \tilde{v}_n(T + 1) = 0,
\]

\[
\omega_n + \tilde{v}_n(k-1) - g_2 \left( k, \frac{1}{n} + \omega_n + \tilde{v}_n \right) + \lambda h_1(k, \bar{v}(k)) + \lambda h_2(k, \omega_n + \tilde{v}_n) \quad \text{for} \quad k \in [1, T].
\]

\[\text{(2.82)}\]
Let
\[ \hat{u}_n(k) = \omega_n(k) + \tilde{v}_n(k) \quad \text{for } k \in [0, T + 1]. \]  
(2.83)

Then, \( \hat{u}_n \in C[0, T + 1] \), \( \hat{u}_n(1) = \hat{u}_n(T + 1) = 0 \).

We let
\[ \hat{u}(k) = \omega(k) + R_0 \phi_1(k) \quad \text{for } k \in [0, T + 1], \]  
(2.84)

and so
\[ 0 \leq \hat{u}_n(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T + 1]. \]  
(2.85)

From Lemma 2.11, we have
\[ -\Delta^2 \bar{u}(k - 1) \leq -g_1(k, \bar{u}(k)) + \lambda h_1(k, \bar{u}(k)) \]
\[ \leq \lambda h_1(k, \bar{u}(k)) \]
\[ \leq \lambda h_1(k, \bar{u}(k)) + g_2 \left( k, \frac{1}{n} + \hat{u}_n(k) \right) + \lambda h_2(k, \hat{u}_n(k)) \]
\[ \leq -\Delta^2 \hat{u}_n(k - 1) \quad \text{for } k \in [1, T], \]
(2.86)

that is,
\[ -\Delta^2 (\bar{u} - \hat{u}_n)(k - 1) \leq 0. \]  
(2.87)

A standard argument (see the argument to show (2.35)) yields
\[ \bar{u}(k) \leq \hat{u}_n(k) \quad \text{for } k \in [1, T]. \]  
(2.88)

Let
\[ a_n(k) = \sup \left\{ \left| \frac{\partial}{\partial s} g_2 \left( k, \frac{1}{n} + s \right) \right| : 0 < s \right\}, \]
(2.89)

and notice that \( s \rightarrow g_2(k, 1/n + s) + a(k)s \) is increasing. Let \( \bar{u}_n = \bar{u} \). From (2.85) and (2.88), we have
\[ \bar{u}(k) = \bar{u}_n(k) \leq \hat{u}_n(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T + 1]. \]  
(2.90)

Also for \( v \in C[1, T] \) with \( \bar{u}_n(k) \leq v(k) \leq \hat{u}_n(k), k \in [1, T] \), we have
\[ -\Delta^2 \bar{u}_n(k - 1) + a_n(k)\bar{u}_n(k) = -g_1(k, \bar{u}_n(k)) + \lambda h_1(k, \bar{u}_n(k)) + a_n(k)\bar{u}_n(k) \]
\[ \leq -g_1(k, v(k)) + \lambda h_1(k, v(k)) + a_n(k)v(k) \]
\[ \leq -g_1 \left( k, \frac{1}{n} + v(k) \right) + \lambda h_1(k, v(k)) + a_n(k)v(k) \]
\[ \leq g_2 \left( k, \frac{1}{n} + v(k) \right) + \lambda h_2(k, v) + a_n(k)v(k) \quad \text{for } k \in [1, T], \]
(2.91)

so (2.14) holds.
Also for \( v \in C[1, T] \) with \( \overline{u}_n(k) \leq v(k) \leq \hat{u}_n(k) \), \( k \in [1, T] \), we have

\[
- \Delta^2 \hat{u}_n(k - 1) + a_n(k) \hat{u}_n(k) \\
\geq g_2 \left( k, \frac{1}{n} + \hat{u}_n(k) \right) + \lambda h_1 \left( k, \overline{u}(k) \right) + \lambda h_2 \left( k, \hat{u}_n(k) \right) + a_n(k) \hat{u}_n(k) \\
\geq g_2 \left( k, \frac{1}{n} + \hat{u}_n(k) \right) + a_n(k) \hat{u}_n(k) + \lambda h_2 \left( k, \hat{u}_n(k) \right) \\
\geq g_2 \left( k, \frac{1}{n} + v(k) \right) + a_n(k) v(k) + \lambda h_2 \left( k, v(k) \right) \\
\geq g \left( k, \frac{1}{n} + v(k) \right) + \lambda h(k, v(k)) + a_n(k) v(k) \quad \text{for } k \in [1, T],
\]

so (2.15) holds. Lemma 2.7 guarantees that there exists a solution \( u \in C[0, T + 1] \) to (1.1) with

\[
\overline{u}(k) \leq u(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T + 1].
\]

Moreover, because \( \hat{u}(k) \leq |\omega|_{\infty} + R_0 \phi_1(k) \leq (|\omega|_{\infty} + R_0)(1 + \phi_1(k)) \) and \( c \phi_1(k) \leq \overline{u}(k) \) (see Lemma 2.11), the estimates asserted in the theorem follow. \( \square \)

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References


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