Let $\rho(x,t)$ denote a family of probability density functions parameterized by time $t$. We show the existence of a family $\{\tau_t : t > 0\}$ of deterministic nonlinear (chaotic) point transformations whose invariant probability density functions are precisely $\rho(x,t)$. In particular, we are interested in the densities that arise from the diffusions. We derive a partial differential equation whose solution yields the family of chaotic maps whose density functions are precisely those of the diffusion.

1. Introduction

In this paper, we establish a method for describing flows of probability density functions by means of discrete-time chaotic maps. We start with a standard map whose invariant probability density function is known and then use it to derive other invariant probability density functions by a simple conjugation process which solves the inverse Perron-Frobenius problem [2, 3] in a time-varying setting.

2. Notation and preliminary results

In this paper, we consider space to consist of 1 dimension although the extension to 2 and 3 dimensions is straightforward. In the sequel, we will need some notions from ergodic theory and nonlinear dynamics, which can be found in [1].

Let $\mathbb{R} = (-\infty, \infty)$ and let $T : \mathbb{R} \rightarrow \mathbb{R}$ possess a unique absolutely continuous invariant measure $\mu$ which has the probability density function $f$, that is,

$$\int_A f \, dx = \int_{T^{-1}A} f \, dx \quad (2.1)$$

for any measurable set $A \subset \mathbb{R}$. The Perron-Frobenius operator $P_T$ acting on the space of integrable functions is defined by

$$\int_A f \, dx = \int_{T^{-1}A} P_T f \, dx. \quad (2.2)$$
The operator $P_T$ transforms probability density functions into probability density functions under the transformation $T$, where $T$ is assumed to be nonsingular. One of the most important properties of $P_T$ is that its fixed points are the densities of measures invariant under $T$ [1].

Let $h : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Then $\tau = h^{-1} \circ T \circ h$ is a transformation from $\mathbb{R}$ into $\mathbb{R}$, which is differentially conjugate to $T$ and whose probability density function is given by

$$k = (f \circ h) \cdot |h'|.$$  \hspace{1cm} (2.3)

We assume that $T$ is a piecewise monotonic expanding $C^1$ map on $\mathbb{R}$ that admits a unique absolutely continuous invariant measure. Then the invariant density function $f(x)$ is a fixed point of the Perron-Frobenius operator $P_T$ [1]. We now consider the inverse Perron-Frobenius problem: suppose we are given a probability density function $g(x)$ on $\mathbb{R}$, can we find a transformation $\tau$ such that $g(x)$ is the unique probability density function invariant under $\tau$? This problem has been dealt with by Ershov and Malinetskiǐ [2] and in [3] from a computational perspective.

We solve the inverse Perron-Frobenius problem by applying (2.3), that is, we find $h$ such that

$$(f \circ h) \cdot h' = g,$$  \hspace{1cm} (2.4)

where we have assumed, without loss of generality, that $h$ is an increasing function on $\mathbb{R}$. Now, let

$$F(x) = \int_{-\infty}^{x} f(y) dy$$  \hspace{1cm} (2.5)

be the distribution function associated with $f$. Then, from (2.4) and the change-of-variable formula, we have

$$F(h(x)) = \int_{-\infty}^{x} g(y) dy.$$  \hspace{1cm} (2.6)

Since $F$ is a monotonically increasing function, it has a unique inverse and

$$h(x) = F^{-1}\left(\int_{-\infty}^{x} g(y) dy\right).$$  \hspace{1cm} (2.7)

Thus, we have found $h(x)$ such that $\tau = h^{-1} \circ T \circ h$ has the probability density function $g(x)$. Summarizing, given any probability density function $g(x)$, we have proven the existence of a point transformation $\tau$ whose probability density function is $g(x)$.

**Example 2.1.** Let

$$T(x) = a \tan x, \quad x \neq \frac{k\pi}{2}, \quad k = \pm 1, \pm 3, \ldots,$$  \hspace{1cm} (2.8)
and $a>1$. Then the probability density function invariant under $T$ is [1]

$$f(x) = \frac{p}{\pi(p^2 + x^2)}, \quad (2.9)$$

where $p > 0$ satisfies the equation $a \tanh(p) = p$. For $p > 2$, $a \approx p$ and we can assume that $T(x) = p \tan x$. Hence, for $p = 4$, say,

$$f(x) = \frac{4}{\pi(16 + x^2)}, \quad (2.10)$$

$$F(x) = \int_{-\infty}^{x} \frac{4}{\pi(16 + y^2)} \, dy = \frac{1}{2} + \frac{1}{\pi} \arctan \left( \frac{x}{4} \right),$$

$$F^{-1}(x) = -4 \cot(\pi x). \quad (2.11)$$

Now, suppose we want to find a map $\tau$ whose unique invariant probability density function is given by $g(x) = (\cos 8x) \exp(-3x^2)$. We obtain

$$\int_{-\infty}^{x} (\cos 8x) \exp(-3x^2) \, dy = \frac{2 \exp(-2x^2)(2/\pi) \cos^2(8x)}{1 + \exp(-64/3)} \quad (2.12)$$

from which we can determine $h(x)$ using (2.7). Once $h$ is known, so is $\tau = h^{-1} \circ T \circ h$, whose probability density function is $g(x)$.

The foregoing method can be extended to a family of probability density functions $\{g_t(y) : t \in I\}$. In this case, the homeomorphism $h$ becomes a family of homeomorphisms $\{h_t : t \in I\}$ parameterized by $t$, where

$$h_t(x) = F^{-1} \left( \int_{-\infty}^{x} g_t(y) \, dy \right). \quad (3.1)$$

3. Chaotic map description of diffusions

Consider the diffusion equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\nabla (v(x,t)\rho(x,t)) = -\frac{\partial}{\partial x} \left[ b(x,t)\rho(x,t) - \frac{1}{2} \frac{\partial \rho(x,t)}{\partial x} \right], \quad (3.2)$$

where $b(x,t)$ is the forward drift coefficient. Our objective is to prove the existence of a family of point transformations $\{\tau_t \in \Gamma : t > 0\}$ whose invariant probability density functions are $\{\rho_t : t > 0\}$. To do this, we let $T$ be the transformation defined by (2.8) and we derive a partial differential equation for $h_t(x)$ such that $\{\tau_t = h_t^{-1} \circ T \circ h_t\}$ possesses $\{\rho(x,t) : t > 0\}$ as the associated family of invariant probability density functions.

Since

$$h_t(x) = F^{-1} \left( \int_{-\infty}^{x} \rho(y,t) \, dy \right), \quad (3.3)$$
140 A description of stochastic systems using chaotic maps

we have

$$\frac{\partial}{\partial t} h_t(x) = (F^{-1})' \left( \int_{-\infty}^{x} \rho(y,t)dy \right) \left( \int_{-\infty}^{x} \frac{\partial}{\partial t} \rho(y,t)dy \right).$$

(3.3)

Noting that

$$(F^{-1})' \left( \int_{-\infty}^{x} \rho(y,t)dy \right) = \frac{1}{F'(F^{-1}(\int_{-\infty}^{x} \rho(y,t)dy))}$$

(3.4)

and using (3.1), we obtain

$$\frac{\partial}{\partial t} h_t(x) = \frac{1}{f(h_t)} \int_{-\infty}^{x} \left[ -\frac{\partial}{\partial y} b(y,t)\rho(y,t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\rho(y,t)) \right] dy. \quad (3.5)$$

Thus,

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = -b(x,t)\rho(x,t) + \frac{1}{2} \frac{\partial}{\partial x} (\rho(x,t)) + q(t), \quad (3.6)$$

where $q(t)$ is an unknown function of $t$ only. To find $q(t)$, we return to (3.3) and write

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = \frac{\partial}{\partial t} \int_{-\infty}^{x} \rho(y,t)dy. \quad (3.7)$$

We assume that $\rho(y,t)$ and $(\partial/\partial x)(\rho(x,t))$ both go to 0 as $x \to \infty$; then the right-hand side of (3.7) also goes to 0 since

$$\int_{-\infty}^{\infty} \rho(y,t)dy = 1 \quad (3.8)$$

for all $t \geq 0$. Hence, $q(t) = 0$ for all $t \geq 0$. Thus, (3.6) reduces to

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = -b(x,t) f(h_t(x)) \frac{\partial}{\partial t} h_t(x) + \frac{1}{2} \frac{\partial}{\partial x} \left( f(h_t(x)) \frac{\partial}{\partial t} h_t(x) \right) \quad (3.9)$$

or

$$\left( 1 + b(x,t) \right) f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = \frac{1}{2} \frac{\partial}{\partial x} \left( f(h_t(x)) \frac{\partial}{\partial t} h_t(x) \right), \quad (3.10)$$

whose solution is the family of homeomorphisms $\{h_t\}$ which determine the family of deterministic chaotic maps $\{\tau_t = h_t^{-1} \circ T \circ h_t\}$, whose probability density functions are equal to $\rho(x,t)$.

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