PERIODIC SOLUTIONS FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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We study the existence of a periodic solution for some partial functional differential equations. We assume that the linear part is nondensely defined and satisfies the Hille-Yosida condition. In the nonhomogeneous linear case, we prove the existence of a periodic solution under the existence of a bounded solution. In the nonlinear case, using a fixed-point theorem concerning set-valued maps, we establish the existence of a periodic solution.

1. Introduction

Consider the partial functional differential equation

\[ \frac{d}{dt}x(t) = Ax(t) + L(t, x_t) + G(t, x_t), \quad \text{for } t \geq 0, \]

\[ x_0 = \varphi \in C = C([-r, 0]; E), \tag{1.1} \]

where \( A : D(A) \subset E \to E \) is a nondensely defined linear operator on a Banach space \( E \). Throughout this paper, we suppose that

(H1) \( A \) is a Hille-Yosida operator: there exist \( M_0 \geq 1 \) and \( \omega_0 \in \mathbb{R} \) such that

\[ (\omega_0, \infty) \subset \rho(A), \quad \| R(\lambda, A) \|^n \leq \frac{M_0}{(\lambda - \omega_0)^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \omega_0, \tag{1.2} \]

where \( \rho(A) \) is the resolvent set of \( A \) and \( R(\lambda, A) = (\lambda - A)^{-1} \).

\( C \) is the space of continuous functions from \([-r, 0]\) into \( E \) endowed with the uniform norm topology, and for every \( t \geq 0 \), the history function \( x_t \in C \) is defined by

\[ x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in [-r, 0]. \tag{1.3} \]

\( L : \mathbb{R} \times C \to E \) is continuous, linear with respect to the second argument and \( \omega \)-periodic in \( t \); \( G : \mathbb{R} \times C \to E \) is continuous and \( \omega \)-periodic in \( t \).

When the operator \( A \) generates a strongly continuous semigroup on \( E \), (1.1) has been treated extensively by several authors; for more details, we refer to [14]. Recently in [1, 8],
the existence, the regularity of solutions, and the local stability have been treated when
A is nondensely defined and satisfies the Hille-Yosida condition. In this work, we will
deal with the existence of periodic solutions of (1.1) when A satisfies the Hille-Yosida
condition. The problem of finding periodic solutions is an important subject in the qual-
itative study of functional differential equations. The famous Massera’s theorem on two-
dimensional periodic ordinary differential equations [11] explains the relationship be-
tween the boundedness of solutions and periodic solutions. In [15], using Browder’s
fixed-point theorem, it has been proved that if the solutions of an n-dimensional peri-
odic ordinary differential equation are either uniformly bounded or uniformly ultimately
bounded, then the system has a periodic solution. In [5], the existence of a periodic solu-
tion has been established under the existence of a bounded solution for some inhomoge-
neous, linear functional differential equation in infinite dimensional space. In [10], using
Horn’s fixed-point theorem, the existence of periodic solutions for functional differential
equation with finite delay was established. Recently in [12], several criteria were obtained
to ensure the existence and uniqueness of a periodic solution for some inhomogeneous
linear partial functional differential equations with infinite delay. In [4], we developed
some results dealing with the existence of a periodic solution for (1.1) when A gener-
ates a strongly continuous semigroup on E. In [7], it was established that the existence
of bounded and ultimate bounded solutions of (1.1) implies the existence of periodic
solutions. The approach that was used was based on Horn’s fixed-point theorem. In this
paper, we generalize the results obtained in [4, 5, 11] for (1.1), where the operator A is
not necessarily densely defined but satisfies the Hille-Yosida condition. In Section 2, we
prove the existence of periodic solutions in the nonhomogeneous linear case under the
assumption that a bounded solution on \( \mathbb{R}^+ \) exists. In Section 3, we study the nonlinear
case; our approach makes use of a fixed-point theorem for set-valued maps to obtain
sufficient conditions, ensuring the existence of a periodic solution for (1.1). Section 4 is
devoted to an example.

2. Inhomogeneous linear case

**Definition 2.1** [1, 8]. A continuous function \( x : [-r, b] \to E \) \((b > 0)\) is called an integral
solution of (1.1) if

(i) \( \int_0^t x(s)ds \in D(A) \), for \( t \in [0, b] \),

(ii) \( x(t) = \varphi(0) + A \int_0^t x(s)ds + \int_0^t L(s,x_s)ds + \int_0^t G(s,x_s)ds \), for \( t \in [0, b] \),

(iii) \( x_0 = \varphi \).

It follows from the closedness of A that if \( x \) is an integral solution of (1.1), then \( x(t) \in \overline{D(A)} \), for \( t \geq 0 \). The following result dealing with the existence and the uniqueness of the
integral solution was established.

**Theorem 2.2** [1, 8]. Assume that (H1) holds and G is Lipschitz with respect to the second
argument. Then for all \( \varphi \in C \) such that \( \varphi(0) \in \overline{D(A)} \), (1.1) has a unique integral solution
on \( \mathbb{R}^+ \). Moreover, the integral solution depends continuously on the initial data.

Let \( A_0 \) be the part of A in \( \overline{D(A)} \) given by

\[
A_0 = A \quad \text{on } D(A_0) = \{ x \in D(A) : Ax \in \overline{D(A)} \}. \tag{2.1}
\]
Then, from [2], $A_0$ generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$. Moreover, from [13], if the integral solution of (1.1) exists, then it is given by this variation of constant formula

$$
\frac{d}{dt} x(t) = Ax(t) + L(t, x_t) + f(t), \quad t \geq 0,
$$

where $B_\lambda = \lambda (\lambda - A)^{-1}$.

Consider the equation

$$
\frac{d}{dt} x(t) = Ax(t) + L(t, x_t) + f(t), \quad t \geq 0,
$$

where $f$ is continuous and $\omega$-periodic in $t$, and suppose the hypothesis stated below.

(H2) The semigroup $(T_0(t))_{t \geq 0}$ is compact on $\overline{D(A)}$, meaning that for $t > 0$, the operator $T_0(t)$ is compact on $\overline{D(A)}$.

**Theorem 2.3.** Assume that (H1) and (H2) hold. Then the following are equivalent:

(i) there exists a $\varphi \in C$ such that (2.3) has a bounded integral solution defined on $\mathbb{R}^+$,

(ii) equation (2.3) has an $\omega$-periodic solution.

Let $u$ be the bounded integral solution of (2.3) on $\mathbb{R}^+$, then the following two lemmas are needed in the proof of Theorem 2.3.

**Lemma 2.4.** {\{u(t) : t \geq 0\} is relatively compact in $E$ and $u$ is uniformly continuous. Consequently, {\{u(t) \neq 0\} is relatively compact in $C$.

**Proof of Lemma 2.4.** For simplicity, we equate $F(t, \varphi) = L(t, \varphi) + f(t)$, and let $\varepsilon > 0$ and $t > \varepsilon$. Then,

$$
u(t) = T_0(t)u(0) + \lim_{\lambda \to \infty} \int_{t-\varepsilon}^{t} T_0(t-s)B_\lambda F(s, u_s) \, ds.
$$

It follows that

$$
u(t) = T_0(t)u(0) + \lim_{\lambda \to \infty} \int_{t-\varepsilon}^{t} T_0(t-s)B_\lambda F(s, u_s) \, ds.
$$

The compactness property of the semigroup $(T_0(t))_{t \geq 0}$ and the boundedness of the solution $u$ show that \{\{T_0(t)u(t-\varepsilon) : t > \varepsilon\} is relatively compact in $E$. Using the boundedness of $B_\lambda$ and $F$, there exists a positive constant $a$ such that

$$
\left\| \lim_{\lambda \to \infty} \int_{t-\varepsilon}^{t} T_0(t-s)B_\lambda F(s, u_s) \, ds \right\| \leq a\varepsilon.
$$

Hence, \{\{u(t) : t \geq 0\} is relatively compact in $E$.}
To show the uniform continuity of \( u \), let \( t > \tau > 0 \). Then,

\[
\begin{align*}
    u(t) - u(\tau) &= (T_0(t) - T_0(\tau))u(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)B_\lambda F(s,u_s)\,ds \\
    &\quad - \lim_{\lambda \to \infty} \int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds.
\end{align*}
\] (2.7)

Since

\[
\begin{align*}
    u(t) - u(\tau) &= (T_0(t-\tau) - I)T_0(\tau)u(0) + (T_0(t-\tau) - I) \lim_{\lambda \to \infty} \int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds \\
    &\quad + \lim_{\lambda \to \infty} \int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds,
\end{align*}
\] (2.8)

we have

\[
\begin{align*}
    u(t) - u(\tau) &= (T_0(t-\tau) - I)u(\tau) + \lim_{\lambda \to \infty} \int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds.
\end{align*}
\] (2.9)

Now the range of \( u \) is relatively compact, so

\[
\lim_{h \to 0} (T_0(h) - I)\xi = 0, \quad \text{uniformly in } \xi \in \{u(t) : t \geq 0\}. \tag{2.10}
\]

Consequently,

\[
\lim_{t-\tau \to 0} \| (T_0(t-\tau) - I)u(\tau) \| = 0. \tag{2.11}
\]

On the other hand, we have

\[
\lim_{t-\tau \to 0} \| \lim_{\lambda \to \infty} \int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds \| = 0. \tag{2.12}
\]

Therefore,

\[
\lim_{t-\tau \to 0} \| u(t) - u(\tau) \| = 0. \tag{2.13}
\]

Using a similar argument, one can also show that

\[
\lim_{t-\tau \to 0} \| u(t) - u(\tau) \| = 0. \tag{2.14}
\]

From the uniform continuity of \( u \), we determine that \( \{u_t : t \geq 0\} \) is an equicontinuous family of functions on \([-r,0]\); moreover, the range of \( u \) is relatively compact. Hence, by Arzela-Ascoli theorem, we determine that \( \{u_t : t \geq 0\} \) is relatively compact in \( C \). \( \square \)

**Lemma 2.5** [9]. Let \( X \) be a Banach space, let \( \Phi : X \to X \) be a continuous linear operator, let \( y \in X \) be given, and let \( \Theta : X \to X \) be given by \( \Theta x = \Phi x + y \). Suppose that there exists \( x_0 \in X \) such that \( \{\Theta^n x_0 : n \in \mathbb{N}\} \) is relatively compact. Then \( \Theta \) has a fixed point.
Proof of Theorem 2.3. As usual, define the Poincaré map $P(\varphi) = x_\omega(\cdot, \varphi, f)$ on the phase space $C_0 = \{\varphi \in C : \varphi(0) \in D(A)\}$, where $x(\cdot, \varphi, f)$ is the integral solution of (2.3). Because of the uniqueness property, it is enough to show that $P$ has a fixed point to get an $\omega$-periodic solution of (2.3). Also, the uniqueness property of the solution with respect to $\varphi$ allows the Poincaré map $P$ to be decomposed as

$$P(\varphi) = x_\omega(\cdot, \varphi, 0) + x_\omega(\cdot, 0, f),$$

where $x_\omega(\cdot, \varphi, 0)$ is the integral solution of (2.3) with $f = 0$, and $x_\omega(\cdot, 0, f)$ is the integral solution of (2.3) with $\varphi = 0$. Let $u$ be the bounded solution of (2.3) on $[0, +\infty)$ and $u_0 = \varphi$. Then, by Lemma 2.4,

$$\{P^n \varphi : n \in \mathbb{N}\} = \{u_{n\omega} : n \in \mathbb{N}\}$$

is relatively compact in $C_0$, and the mapping $P$ has a fixed point in $C_0$ using Lemma 2.5. Hence, (2.3) has an $\omega$-periodic solution.

3. Nonlinear case

Consider the nonlinear equation

$$\frac{d}{dt}x(t) = Ax(t) + L(t, x_t) + G(t, x_t), \quad t \geq 0,$$

and assume the hypothesis stated below.

(H3) $G$ takes every bounded set into a bounded set.

Let $B_\omega$ be the space of all continuous $\omega$-periodic functions from $\mathbb{R}^+$ into $E$, endowed with the uniform norm topology.

Theorem 3.1. Assume that (H1), (H2), and (H3) hold. Further, assume that there exists a positive $\rho$ such that for any $y \in S_\rho = \{v \in B_\omega : \|v\| \leq \rho\}$, the equation

$$\frac{d}{dt}x(t) = Ax(t) + L(t, x_t) + G(t, x_t), \quad t \in \mathbb{R}^+,$$

has an $\omega$-periodic integral solution in $S_\rho$. Then, (3.1) has an integral $\omega$-periodic solution on $\mathbb{R}^+$.

For the proof, we need the following definition and theorem.

Definition 3.2 (see [16, Definition 9.3]). Let $\mathcal{G} : M \to 2^M$ be a multivalued map, where $M$ is a subset of a Banach space and $2^M$ is the power set of $M$.

(i) For $D \subset M$, the inverse image $\mathcal{G}^{-1}(D)$ is the set of all $x \in M$ such that $\mathcal{G}(x) \cap D \neq \emptyset$.

(ii) The map $\mathcal{G}$ is called upper semicontinuous if $\mathcal{G}^{-1}(D)$ is closed for all closed set $D$ in $M$.
Theorem 3.3 (see [16, Corollary 9.8]). Let \( \mathcal{G} : M \rightarrow 2^{\mathbb{M}} \) be a multivalued map, where \( M \) is a nonempty convex set in the Banach space \( X \) such that

(i) the set \( \mathcal{G}(x) \) is nonempty, closed, and convex for all \( x \in M \),
(ii) the set \( \mathcal{G}(M) \) is relatively compact,
(iii) the map \( \mathcal{G} : M \rightarrow 2^{\mathbb{M}} \) is upper semicontinuous.

Then \( \mathcal{G} \) has a fixed point in the sense that there exists \( x \in M \) such that \( x \in \mathcal{G}(x) \).

Proof of Theorem 3.1. Define the set-valued mapping \( \mathcal{G} : S_\rho \rightarrow 2^{S_\rho} \), for \( y \in S_\rho \), by

\[
\mathcal{G}(y) = \left\{ x \in S_\rho : x(t) = T_0(t)x(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds, \ t \geq 0 \right\}.
\]

We will show that the mapping \( \mathcal{G} \) satisfies the conditions of Theorem 3.3.

(i) Let \( y \in S_\rho \), \( x_1, x_2 \in \mathcal{G}(y) \), and \( \lambda \in [0,1] \). Then, \( \lambda x_1 + (1 - \lambda)x_2 \in \mathcal{G}(y) \), which implies that \( \mathcal{G}(y) \) is convex. From the continuity of \( L \) and \( G \), we obtain that \( \mathcal{G}(y) \) is a closed set.

(ii) Let \( x \in \mathcal{G}(S_\rho) \), then there exists \( y \in S_\rho \) such that

\[
x(t) = T_0(t)x(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds, \ t \geq 0.
\]

We first show that \( \{x(t) : x \in \mathcal{G}(S_\rho)\} \) is relatively compact in \( E \). Let \( t > 0 \) and \( \varepsilon > 0 \) such that \( t > \varepsilon \). Then,

\[
x(t) = T_0(t)x(0) + T_0(\varepsilon)\lim_{\lambda \to \infty} \int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds
\]

\[
+ \lim_{\lambda \to \infty} \int_{t-\varepsilon}^t T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds.
\]

From the boundedness of \( L \), \( G \) and (H2), we deduce that

\[
\left\{ T(\varepsilon)\lim_{\lambda \to \infty} \int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds : x \in \mathcal{G}(S_\rho) \right\}
\]

is relatively compact in \( E \). On the other hand, for some positive constant \( b \), we have

\[
\lim_{\lambda \to \infty} \int_{t-\varepsilon}^t T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds \leq b\varepsilon, \ \forall x \in \mathcal{G}(S_\rho).
\]

Hence, \( \{x(t) : x \in \mathcal{G}(S_\rho)\} \) is relatively compact in \( E \), for every \( t > 0 \), and by periodicity, we also have that \( \{x(0) : x \in \mathcal{G}(S_\rho)\} \) is relatively compact in \( E \). For the equicontinuity, one has, for \( t > \tau > 0 \),

\[
\|x(t) - x(\tau)\| \leq \|T_0(t) - T_0(\tau)\|\rho + \lim_{\lambda \to \infty} \int_\tau^t T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds
\]

\[
+ \|T_0(t-\tau) - I\| \lim_{\lambda \to \infty} \int_0^\tau T_0(\tau-s)B_\lambda(L(s,x_s) + G(s,y_s))\,ds.
\]
The semigroup \((T_0(t))_{t \geq 0}\) is compact, so \((T_0(t))_{t \geq 0}\) is continuous in the uniform topology whenever \(t > 0\). Hence,

\[
\lim_{t \to \tau} \| T_0(t) - T_0(\tau) \| = 0. \tag{3.9}
\]

By \((H_3)\), we deduce that for some positive constant \(c\),

\[
\int_{\tau}^{t} \| T_0(t - s)B_{\lambda}(L(s,x_s) + G(s,y_s)) \| ds \leq c(t - \tau), \quad \text{uniformly for } x, y \in S_{\rho}. \tag{3.10}
\]

Since \(\{x(t) : x \in \mathcal{G}(S_{\rho})\}\) is relatively compact in \(E\) for every \(t > 0\), \(\{x(t) - T(t)x(0) : x \in \mathcal{G}(S_{\rho})\}\) is also relatively compact in \(E\). Moreover, there exists a compact set \(K\) in \(E\) such that

\[
\lim_{\lambda \to -\infty} \int_{0}^{T} T_0(\tau - s)B_{\lambda}(L(s,x_s) + G(s,y_s)) ds \in K, \quad \forall x \in \mathcal{G}(S_{\rho}). \tag{3.11}
\]

Consequently,

\[
\lim_{h \to 0} (T_0(h) - I) \xi = 0, \quad \text{uniformly in } \xi \in K, \quad \lim_{t \to \tau} \sup_{x \in \mathcal{G}(S_{\rho})} \| x(t) - x(\tau) \| = 0. \tag{3.12}
\]

Similarly, one can also prove that

\[
\lim_{t \to \tau} \sup_{x \in \mathcal{G}(S_{\rho})} \| x(t) - x(\tau) \| = 0. \tag{3.13}
\]

Therefore, \(\mathcal{G}(S_{\rho})\) is a family of uniformly bounded and equicontinuous \(\omega\)-periodic functions. By the Arzèla-Ascoli theorem, we deduce that \(\mathcal{G}(S_{\rho})\) is relatively compact in \(B_{\omega}\).

(iii) To prove that \(\mathcal{G}\) is upper semicontinuous, it is enough to show that \(\mathcal{G}\) is closed. Let \((y^n)_{n \geq 0}\) and \((z^n)_{n \geq 0}\) be sequences, respectively, in \(S_{\rho}\) and \(\mathcal{G}(S_{\rho})\) such that

\[
y^n \to y, \quad z^n \to z \quad \text{as } n \to \infty, \quad z^n \in \mathcal{G}(y^n), \quad \forall n \geq 0. \tag{3.14}
\]

Then,

\[
z^n(t) = T_0(t)z^n(0) + \lim_{\lambda \to -\infty} \int_{0}^{t} T_0(t - s)B_{\lambda}(L(s,z^n_s) + G(s,y^n_s)) ds, \quad t \geq 0. \tag{3.15}
\]

Letting \(n\) go to infinity and by a continuity argument, we obtain

\[
z(t) = T_0(t)z(0) + \lim_{\lambda \to -\infty} \int_{0}^{t} T_0(t - s)B_{\lambda}(L(s,z_s) + G(s,y_s)) ds, \quad t \geq 0. \tag{3.16}
\]

Hence, \(z \in \mathcal{G}(y)\), which implies that \(\mathcal{G}\) is closed. Now let \(D\) be a closed set in \(S_{\rho}\) and take a sequence \((x_n)_{n \in \mathcal{G}^{-1}(D)}\) such that \(x_n \to x\) as \(n \to \infty\). Since \(x_n \in \mathcal{G}^{-1}(D)\), it follows that there exists \(y_n \in D\) such that \(y_n \in \mathcal{G}(x_n)\). Moreover, \(\mathcal{G}(S_{\rho})\) is compact; thus, there exists a subsequence \((y'_n)_{n}\) of \((y_n)_{n}\) such that \(y'_n \to y\) as \(n \to \infty\). Therefore, \(\mathcal{G}\) is closed and it
follows that \( y \in \mathcal{G}(x) \) and \( y \in \mathcal{G}^{-1}(D) \). Consequently, \( \mathcal{G} \) is upper semicontinuous. All the assumptions of Theorem 3.3 hold. Hence, there exists \( x \in S_\rho \) such that \( x \in \mathcal{G}(x) \). Finally, \( x \) is an \( \omega \)-periodic solution of (3.1) on \( \mathbb{R}^+ \).

To prove that (3.2) has an \( \omega \)-periodic solution in \( S_\rho \), it suffices, by Theorem 2.3, to show that it has a solution which is bounded by \( \rho \).

**Corollary 3.4.** Assume that (H1), (H2), and (H3) hold. If there exists a positive \( \rho \) such that for any \( y \in S_\rho = \{ v \in B_\omega : \| v \| \leq \rho \} \), the nonhomogeneous linear equation (3.2) has an integral solution that is bounded by \( \rho \). Hence, there exists a \( \omega \)-periodic solution on \( \mathbb{R}^+ \) that is bounded by \( \rho \).

**Proof.** Let \( u \) be a bounded solution of (3.2) such that \( u_0 = \phi \). Following the proof of [9, Theorem 2.5], the map \( P \) has a fixed point which belongs to \( \overline{\mathcal{C}} \{ P^n \phi : n \geq 0 \} \), where \( \overline{\mathcal{C}} \) denotes the closure of the convex hull. Let \( \psi \) be the fixed point of \( P \) and \( x(\cdot, \psi, f) \) the associated integral solution; by virtue of the continuous dependence on the initial data, the solution \( x(\cdot, \psi, f) \) is also bounded by \( \rho \). □

**4. Application**

To apply the previous results, we consider the partial differential equation with delay:

\[
\frac{\partial}{\partial t} w(t,x) = \frac{\partial^2}{\partial x^2} w(t,x) + b_1(t)w(t-r,x) + b_2(t)h(w(t-r,x)) + g(t,x), \quad t \geq 0, \; x \in [0,\pi],
\]

\[
w(t,0) = w(t,\pi) = 0, \quad t \geq 0,
\]

\[
w(\theta,x) = \phi(\theta,x), \quad \theta \in [-r,0], \; x \in [0,\pi],
\]

(4.1)

where \( b_1, b_2 : \mathbb{R}^+ \to \mathbb{R} \) are continuous and \( \omega \)-periodic, \( h : \mathbb{R} \to \mathbb{R} \) is continuous such that

\[
|h(x)| \leq k|x|, \quad x \in \mathbb{R},
\]

(4.2)

\( g : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R} \) is continuous and \( \omega \)-periodic in \( t \), and \( \phi : [-r,0] \times [0,\pi] \to \mathbb{R} \) is continuous. Let \( Y = C([0,\pi];\mathbb{R}) \) and \( \Delta \) the Laplacian operator on \([0,\pi]\) with domain

\[
D(\Delta) = \{ z \in C([0,\pi];\mathbb{R}) : \Delta z \in C([0,\pi];\mathbb{R}), \; z(0) = z(\pi) = 0 \}.
\]

(4.3)

Then, by [6], \( \Delta \) satisfies the Hille-Yosida condition in \( Y \); more precisely, one has

\[
(0,\infty) \subset \rho(\Delta), \quad \| R(\lambda,\Delta) \| \leq \frac{1}{\lambda}, \quad \text{for } \lambda > 0.
\]

(4.4)

Moreover,

\[
\overline{D(\Delta)} = \{ z \in C([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0 \} = C_0([0,\pi];\mathbb{R}).
\]

(4.5)

Let \( \Delta_0 \) be the part of \( \Delta \) in \( \overline{D(\Delta)} \) given by

\[
D(\Delta_0) = \{ z \in C_0([0,\pi];\mathbb{R}) : \Delta z \in C_0([0,\pi];\mathbb{R}) \}, \quad \Delta_0 z = \Delta z.
\]

(4.6)

Then, by [3], \( \Delta_0 \) generates a compact semigroup \( (T_0(t))_{t \geq 0} \) on \( C_0([0,\pi];\mathbb{R}) \) such that

\[
\| T_0(t) \| \leq e^{-t}, \quad t \geq 0.
\]

(4.7)
Let $L, G : \mathbb{R} \times C([-r, 0]; Y) \to Y$ be defined, for $t \in \mathbb{R}^+$, $\varphi \in C([-r, 0]; Y)$, and $x \in [0, \pi]$, by

$$
(L(t, \varphi))(x) = b_1(t)\varphi(-r)(x),
$$
$$
(G(t, \varphi))(x) = b_2(t)h(\varphi(-r)(x)) + g(t, x).
$$

Then, (4.1) takes the abstract form

$$
\frac{d}{dt}x(t) = \Delta x(t) + L(t, x_t) + G(t, x_t), \quad \text{for } t \geq 0.
$$

Hence, (H₁), (H₂), and (H₃) are satisfied, and we have the following proposition.

**Proposition 4.1.** Assume that there exists $d \in (0, 1)$ such that

$$
|b_1(t)| + |b_2(t)| k \leq 1 - d, \quad \text{for } t \in [0, \omega].
$$

Then, (4.9) has an $\omega$-periodic solution.

**Proof.** Let $m = \max_{t \in [0, \omega], x \in [0, \pi]} |g(t, x)|$ and $\rho = 1 + m/d$. We claim that if $y$ is a continuous $\omega$-periodic function such that $\|y\| \leq \rho$, then for all $\varphi$ with $\|\varphi\| < \rho$, the solution $x$ of

$$
\frac{d}{dt}x(t) = \Delta x(t) + L(t, x_t) + G(t, y_t), \quad \text{for } t \geq 0,
$$

$$
x_0 = \varphi \in C([-r, 0]; Y),
$$

satisfies $\|x(t)\| \leq \rho$, for all $t \geq 0$. Proceeding by contradiction, suppose that there exists $t_1$ such that $\|x(t_1)\| > \rho$ and let

$$
t_0 = \inf \{t > 0 : \|x(t)\| > \rho \}.
$$

By continuity, we get $\|x(t_0)\| = \rho$ and there exists $\delta > 0$ such that $\|x(t)\| > \rho$, for $t \in (t_0, t_0 + \delta)$. By using the variation of constant formula (2.2),

$$
x(t_0) = T_0(t_0)\varphi(0) + \lim_{\lambda \to -\infty} \int_0^{t_0} T_0(t_0 - s) B_\lambda(L(s, x_s) + G(s, y_s))ds, \quad t \geq 0.
$$

By (4.8), we get that

$$
\|x(t_0)\| \leq e^{-t_0} \rho + ((|b_1| + |b_2| k) \rho + m)(1 - e^{-t_0}),
$$

and by condition (4.10), we obtain

$$
\|x(t_0)\| \leq \rho + (m - \rho d)(1 - e^{-t_0})
$$

or $\|x(t_0)\| \leq \rho - d(1 - e^{-t_0})$, which gives that $\|x(t_0)\| < \rho$. This contradicts the definition of $t_0$. Consequently, $\|x(t)\| \leq \rho$ for all $t \geq 0$, and by Corollary 3.4, (4.9) has an $\omega$-periodic solution in $S_\rho$.  \qed
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References


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