EXISTENCE OF SOLUTIONS OF
SOBOLEV-TYPE SEMILINEAR MIXED
INTEGRODIFFERENTIAL INCLUSIONS
IN BANACH SPACES

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The existence of mild solutions of Sobolev-type semilinear mixed integrodifferential inclusions in Banach spaces is proved using a fixed point theorem for multivalued maps on locally convex topological spaces.

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1 Introduction

The problem of proving the existence of mild solutions for differential and integrodifferential equations in abstract spaces has been studied by several authors [2, 4, 11, 12, 13]. Balachandran and Uchiyama [3] established the existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. Benchohra [6] studied the existence of mild solutions on infinite intervals for a class of differential inclusions in Banach spaces. For the existence results of differential inclusions on compact intervals, one can refer to the papers of Avgerinos and Papageorgiou [1], and Papageorgiou [14, 15]. Benchohra and Ntouyas [7] discussed the existence results for first order integrodifferential inclusions of the form

\[
\frac{dy}{dt} - Ay \in F(t, \int_0^t k(t, s, y)ds) \quad t \in I = [0, \infty),
\]

\[
y(0) = y_0.
\]

In this paper, we consider the Sobolev-type semilinear mixed integrodifferential inclusion of the type

\[
(Eu(t))' + Au \in G \left(t, u, \int_0^t k(t, s, u)ds, \int_0^a b(t, s, u)ds\right) \quad t \in I = [0, \infty), \quad (1.1)
\]
where \( G : I \times X \times X \times X \to 2^Y \) is a bounded, closed, convex, multivalued map \( k : \Delta \times X \to X \), \( b : \Delta \times X \to X \), where \( \Delta = \{(t, s) \in I \times I; t \geq s\} \), \( u_0 \in X \), \( a \) is a real constant, \( X, Y \) are real Banach spaces with norms \( |||.|\| \) and \( |.| \), respectively. Our method is to reduce the problem (1.1) to a fixed point problem of a suitable multivalued map in the Frechet space \( C(I, X) \) and we make use of a fixed point theorem due to Ma [10] for multivalued maps in locally convex topological spaces.

2 Preliminaries

In this section we introduce the notations, definitions and preliminary facts from multivalued analysis which are used in this paper. \( I_m \) is the compact interval \([0, m] (m \in \mathbb{N})\). \( C(I, X) \) is the linear metric Frechet space of continuous functions from \( I \) into \( X \) with the metric

\[
d(u, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|u - z\|_m}{1 + \|u - z\|_m}
\]

for each \( u, z \in C(I, X) \),

where \( \|u\|_m = \sup\{\|u(t)\| : t \in I_m\} \). \( B(X) \) denotes the Banach space of bounded linear operators from \( X \) into \( X \). A measurable function \( u : I \to X \) is Bochner integrable if and only if \( |u| \) is Lebesgue integrable. Let \( L^1(I, X) \) denote the Banach space of continuous functions \( u : I \to X \) which are Bochner integrable normed by

\[
\|u\|_{L^1} = \int_0^{\infty} \|u(t)\| dt,
\]

and \( U_r \) is a neighbourhood of \( 0 \) in \( C(I, X) \) defined by

\[
U_r = \{ u \in C(I, X) : \|u\|_m \leq r \}
\]

for each \( m \in \mathbb{N} \). The convergence in \( C(I, X) \) is the uniform convergence on compact intervals, that is, \( u_j \to u \) in \( C(I, X) \) if and only if for each \( m \in \mathbb{N} \), \( \|u_j - u\|_m \to 0 \) in \( C(I_m, X) \) as \( j \to \infty \). \( BCC(X) \) denotes the set of all nonempty bounded, closed, and convex subsets of \( X \).

A multivalued map \( G : X \to 2^X \) is convex(closed) valued if \( G(x) \) is convex(closed) for all \( x \in X \). \( G \) is bounded on bounded sets if \( G(B) = \bigcup_{x \in B} G(x) \) is bounded in \( X \) for any bounded set \( B \) of \( X \) (that is, \( \sup_{x \in B} \{\|u\| : u \in G(x)\} < \infty \)). \( G \) is called upper semi continuous on \( X \) if for each \( x_0 \in X \) the set \( G(x_0) \) is a nonempty, closed subset of \( X \), and if for each open subset \( B \) of \( X \) containing \( G(x_0) \), there exists an open neighbourhood \( A \) of \( x_0 \) such that \( G(A) \subseteq B \). \( G \) is said to be completely continuous if \( G(B) \) is relatively compact for every bounded subset \( B \subseteq X \). If the multivalued map \( G \) is completely continuous with nonempty compact values, then \( G \) is upper semicontinuous if and only if \( G \) has a closed graph (that is, \( x_n \to x_0, u_n \to u_0, u_n \in Gx_n \) imply \( u_0 \in Gx_0 \)).

We assume the following conditions:

(i) The operator \( A : D(A) \subset X \to Y \) and \( E : D(E) \subset X \to Y \) satisfy the following conditions
Existence of Solutions of Sobolev-Type

\[ [C_1] \quad A \text{ and } E \text{ are closed linear operators.} \]
\[ [C_2] \quad D(E) \subset D(A) \text{ and } E \text{ is bijective.} \]
\[ [C_3] \quad E^{-1} : Y \rightarrow D(E) \text{ is continuous.} \]
\[ [C_4] \quad \text{The resolvent } R(\lambda, -AE^{-1}) \text{ is a compact operator for some } \lambda \in \rho(-AE^{-1}) \text{ and resolvent set of } -AE^{-1}. \]

Conditions \([C_1], [C_2] \), and the closed graph theorem imply the boundedness of the linear operator \( AE^{-1} : Y \rightarrow Y \).

(ii) \( G : I \times X \times X \times X \rightarrow BCC(Y) \) is measurable with respect to \( t \) for each \( u \in X \), upper semi continuous with respect to \( u \) for each \( t \in I \), and for each \( u \in C(I,X) \) the set
\[
S_{G,u} = \{ g \in L^1(I;R) : g(t) \in G(t,u, \int_0^t k(t,s,u)ds, \int_0^a b(t,s,u)ds) \}
\]
for a.e \( t \in I \) is nonempty.

(iii) There exist functions \( p(t), q(t) \in C(I;R) \) such that
\[
| \int_0^t k(t,s,u)ds | \leq p(t)\|u\| \text{ and } | \int_0^a b(t,s,u)ds | \leq q(t)\|u\| \text{ for a.e } t, s, \in I, u \in X.
\]

(iv) There exists a function \( \alpha(t) \in L^1(I;R^+) \) such that
\[
\|G(t,u,v,w)\| \leq \alpha(t)\Omega(\|u\| + \|v\| + \|w\|)
\]
for a.e \( t \in I, u \in X \), where \( \Omega : R_+ \rightarrow (0,\infty) \) is continuous increasing function satisfying \( \Omega(p(t)x + q(t)y) \leq p(t)\Omega(x) + q(t)\Omega(y) \) and
\[
M \int_0^m \alpha(s)(1 + p(s) + q(s))ds < \int_c^\infty \frac{du}{\Omega(u)}
\]
for each \( m \in N \), where \( c = \|E^{-1}\|M\|Eu_0\| \) and \( M = \max\{\||T(t)||; t \in I\}\).

(v) For each neighbourhood \( U_r \) of 0, \( u \in U_r \) and \( t \in I \), the set
\[
\{ E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)g(s)ds, \ g \in S_{G,u} \}
\]
is relatively compact.

**Definition 2.1:** A continuous function \( u(t) \) of the integral inclusion
\[
u(t) \in E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)G(s,u,\int_0^s k(s,\tau,u(\tau))d\tau, \int_0^a b(s,\tau,u(\tau))d\tau)ds
\]
is called a mild solution of (1.1) on \( I \).

**Lemma 2.1:** \([9]\). Let \( I \) be a compact real interval and let \( X \) be a Banach space. Let \( G \) be a multivalued map satisfying (i) and let \( \Gamma \) be a linear continuous mapping from \( L^1(I,X) \) to \( C(I,X) \). Then the operator
\[
\Gamma \circ S_G : C(I,X) \rightarrow X, \ (\Gamma \circ S_G)(y) = \Gamma(S_G,y)
\]
is a closed graph operator in \( C(I, X) \times C(I, X) \).

**Lemma 2.2:** [10]. Let \( X \) be a locally convex space. Let \( N : X \to X \) be a compact, convex valued, upper semicontinuous, multivalued map such that there exists a closed neighbourhood \( U_r \) of \( 0 \) for which \( N(U_r) \) is a relatively compact set for each \( r \in N \). If the set \( \zeta = \{ y \in X : \lambda y \in N(y) \} \) for some \( \lambda > 1 \) is bounded, then \( N \) has a fixed point.

**Remark:** [9]. If \( \dim X < \infty \) and \( I \) is a compact real interval, then for each \( u \in C(I, X) \), \( S_G,u \) is nonempty.

**Lemma 2.3:** [16]. Let \( S(t) \) be a uniformly continuous semigroup and let \( A \) be its infinitesimal generator. If the resolvent set \( R(\lambda : A) \) of \( A \) is compact for every \( \lambda \in \rho(A) \), then \( S(t) \) is a compact semigroup.

From the above fact, \(-AE^{-1}\) generates a compact semigroup \( T(t) \) in \( Y \). Thus, \( \max_{t \in I} |T(t)| \) is finite and so denote \( M = \max_{t \in I} |T(t)| \).

### 3 Main Result

**Theorem 3.1:** If the assumptions (i)–(v) are satisfied, then the initial value problem (1.1) has at least one mild solution on \( I \).

**Proof:** A solution to (1.1) is a fixed point for the multivalued map \( N : C(I, X) \to 2^{C(I, X)} \) defined by

\[
N(u) = \{ h \in C(I, X) : h(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)g(s)ds, \; g \in S_G,u \},
\]

where

\[
S_G,u = \{ g \in L^1(I, X) : g(t) \in G(t, u, \int_0^t k(t, s, u(s))ds, \int_0^a b(t, s, u(s))ds) \}
\]

for a.e. \( t \in I \).

First we shall prove \( N(u) \) is convex for each \( u \in C(I, X) \). Let \( h_1, h_2 \in N(u) \), then there exist \( g_1, g_2 \in S_G,u \) such that

\[
h_i(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)g_i(s)ds, i = 1, 2, t \in I
\]

Let \( 0 \leq k_1 \leq 1 \), then for each \( t \in I \) we have

\[
(k_1h_1 + (1 - k_1)h_2)t = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t-s)(k_1g_1(s) + (1 - k_1)g_2(s))ds.
\]

Since \( S_G,u \) is convex, thus \( kh_1 + (1 - k)h_2 \in N(u) \). Hence, \( N(u) \) is convex for each \( u \in C(I, X) \).

Let \( U_r = \{ u \in C(I, X) : \| u \| \leq r \} \) be a neighbourhood of \( 0 \) in \( C(I, X) \) and \( u \in U_r \). Then for each \( h \in N(u) \) there exists \( g \in S_G,u \) such that for \( t \in I \), we have

\[
\| h(t) \| \leq \| E^{-1} \| \| T(t) \| \| Eu_0 \| + \int_0^t \| E^{-1} \| \| T(t-s) \| \| g(s) \| ds
\]

\[
\leq \| E^{-1} \| M \| Eu_0 \| + \| E^{-1} \| M \int_0^t \alpha(s)\Omega(\| u \| + p(t)\| u \| + q(t)\| u \|)ds
\]
Consider the functions $u$, We must also prove that there exists $g$ a completely continuous multivalued map. Next we shall prove that $h \in \mathcal{H}$ where $h \in \mathcal{H}$ with $h \in \mathcal{H}$ $t+1$. That is,

$$
\| h(t_1) - h(t_2) \| \leq \| E^{-1}\| (T(t_2) - T(t_1))Eu_0\|
+ \| E^{-1}\| \int_0^{t_2} (T(t_2) - s)g(s)ds \| T(t_1 - s))g(s)ds \|
+ \| E^{-1}\| \int_{t_1}^{t_2} T(t_1 - s)g(s)ds \|
\leq \| E^{-1}\| (T(t_2) - T(t_1))Eu_0\|
+ \| E^{-1}\| \int_0^{t_2} (T(t_2) - s)g(s)ds \|
+ \| M(t_2 - t_1)\| E^{-1}\| \int_0^{t_2} g(s)ds.
$$

Hence, $N(U_r)$ is bounded in $C(I, X)$ for each $r \in N$. Next we shall prove $N(U_r)$ is an equicontinuous set in $C(I, X)$ for each $r \in N$. Let $t_1, t_2 \in I_m$ with $t_1 < t_2$. Then for all $h \in N(u)$ with $u \in U_r$, we have

$$
h_n(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_n(s)ds, \quad t \in I.
$$

We must also prove that there exists $g_0 \in S_{G,u}$ such that

$$
h_0(t) = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_0(s)ds, \quad t \in J. \quad (3.1)
$$

To prove the above, we use the fact that $h_n \to h_0$; and $h_n - E^{-1}T(t)Eu_0 \in \Gamma(S_{G,u})$, where

$$
(\Gamma_2)(t) = \int_0^t E^{-1}T(t - s)g(s)ds, \quad t \in I.
$$

Consider the functions $u_n, h_n - E^{-1}T(t)Eu_0$ and $g_n$ defined on the interval $[k, k+1]$ for any $k \in N \cup \{0\}$. Then using Lemma 2.1, we can conclude (3.1) is true on the compact interval $[k, k+1]$. That is,

$$
[h_0(t)]_{[k,k+1]} = E^{-1}T(t)Eu_0 + \int_0^t E^{-1}T(t - s)g_0^k(s)ds
$$

for a suitable $L^1$-selection $g_0^k$ of $G(t, u, \int_0^t k(t, s, u)ds, \int_0^T b(t, s, u)ds)$ on the interval $[k, k+1]$. Let $g_0(t) = g_0^k(t)$ for $t \in [k, k+1]$. Then $g_0$ is an $L^1$-selection and (3.1)
will satisfied. Clearly we have \( \|(h_n - E^{-1}T(t)Eu_0) - (h_0 - E^{-1}T(t)Eu_0)\|_\infty \to 0 \) as \( n \to \infty \). Consider for all \( k \in N \cup \{0\} \), the mapping

\[
S^k_G : C([k, k+1], X) \to L^1([k, k+1], X),
\]

\[
y \to S^k_{G,y} = \{g \in L^1([k, k+1], X) : g(t) \in G(t, u, \int_0^t k(t, s, u)ds, \int_0^a b(t, s, u)ds) \text{ for } a.e \ t \in [k, k+1]\}.
\]

Moreover, we have \( \|\rangle \|_{\infty} \to \infty \) for \( m \to \infty \).

From Lemma 2.1 it follows that \( \Gamma_k \) is a closed graph operator for all \( k \in N \cup \{0\} \). Moreover, we have

\[
\langle h_n(t) - E^{-1}T(t)Eu_0 \rangle_{[k, k+1]} \in \Gamma_k(S^k_{G,u_n}),
\]

and \( u_n \to u_* \). From Lemma 2.1, we have \( \langle h_0(t) - E^{-1}T(t)Eu_0 \rangle_{[k, k+1]} \in \Gamma_k(S^k_{G,u_*}) \),

\[
\langle h_0(t) - E^{-1}T(t)Eu_0 \rangle_{[k, k+1]} = \int_0^t E^{-1}T(t - s)g(s)ds.
\]

Hence, the function \( g_0 \) defined on \( I \) by \( g_0(t) = g^k_0(t) \) for \( t \in [k, k+1] \) is in \( S^k_{G,u_*} \). Therefore, \( N(U_r) \) is relatively compact for each \( r \in N \) where \( N \) is upper semicontinuous with convex closed values. Finally we prove the set \( \zeta = \{u \in C(I, X); \lambda u \in Nu\} \), for some \( \lambda > 1 \), is bounded.

Let \( \lambda u = Nu \) for some \( \lambda > 1 \). Then there exists \( g \in S^k_{G,u} \) such that

\[
u(t) = \lambda^{-1}E^{-1}T(t)Eu_0 + \lambda^{-1} \int_0^t E^{-1}T(t - s)g(s)ds, \ t \in I,
\]

\[
\|\nu(t)\| \leq \|E^{-1}\|M\|Eu_0\| + \|E^{-1}\|M \int_0^t \alpha(s)(1 + p(s) + q(s))\Omega(\|u\|)ds.
\]

Let \( \nu(t) = \|E^{-1}\|M\|Eu_0\| + \|E^{-1}\|M \int_0^t \alpha(s)(1 + p(s) + q(s))\Omega(\|u\|)ds. \) Then we have \( \nu(0) = \|E^{-1}\|M\|Eu_0\| = c \) and \( \|\nu(t)\| \leq \nu(t), t \in I_m \). Using the increasing character of \( \Omega \) we get

\[
\nu'(t) \leq \|E^{-1}\|M\alpha(t)(1 + p(t) + q(t))\Omega(\nu(t)), \ t \in I_m.
\]

The above proves that for each \( t \in I_m \),

\[
\int_{\nu(0)}^{\nu(t)} \frac{du}{\Omega(u)} \leq \|E^{-1}\|M \int_0^m \alpha(s)(1 + p(s) + q(s))ds < \int_0^\infty \frac{du}{\Omega(u)}.
\]

The above inequality implies that there exists a constant \( M_0 \) such that \( \nu(t) \leq M_0, t \in I_m \), and hence that \( \|u\|_{\infty} \leq M_0 \) where \( M_0 \) depends on \( m \) and on the functions \( \alpha, p, \Omega \). Hence, \( \zeta \) is bounded. Thus by Lemma 2.2, \( N \) has a fixed point that is a mild solution of (1.1).
4 Nonlocal Initial Conditions

Several authors have studied the nonlocal Cauchy problem in abstract spaces [2, 3, 4, 11, 12, 13]. The importance of nonlocal conditions is discussed in [4, 5]. In this section we consider a first order Sobolev-type, semilinear, mixed, integrodifferential inclusion (1.1) with the nonlocal initial condition

$$u(0) + f(u) = u_0 \quad (4.1)$$

In addition to the five assumptions in Section 2, we also assume the following.

(vi) $f : C(I, X) \to X$ is a continuous function, and there exists a constant $L > 0$ such that $\|f(u)\| \leq L$ for each $u \in X$.

(vii) $\|E^{-1}\| M \int_0^\alpha \alpha(s)(1 + p(s) + q(s))ds < \int_1^\infty \frac{du}{\in(x)}$ where $c_1 = \|E^{-1}\| M|Eu_0| + L\|E^{-1}\| M|Eu_0|$.

(viii) For each neighbourhood $U_r$ of 0, $u \in U_r$ and $t \in I$, the set $\{E^{-1}T(t)Eu_0 - E^{-1}T(t)Ef(u) + \int_0^t \int_0^s T(t-s)G(s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^s b(s, \tau, u(\tau))d\tau)ds, g \in S_G, u \}$ is relatively compact.

**Definition 4.1:** A continuous function $u(t)$ of the integral inclusion

$$u(t) \in E^{-1}T(t)Eu_0 - E^{-1}T(t)Ef(u) + \int_0^t E^{-1}T(t-s)G(s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^s b(s, \tau, u(\tau))d\tau)ds$$

is called a mild solution of (1.1)-(4.1) on $I$.

**Theorem 4.1:** If the assumptions (i)–(iii), (vi)–(viii) are satisfied, then the nonlocal initial value problem (1.1)–(4.1) has at least one mild solution on $I$.

The proof of Theorem 4.1 is similar to Theorem 3.1 and hence, is omitted.

**References**


