SMOOTHNESS OF INVARIANT DENSITY FOR EXPANDING TRANSFORMATIONS IN HIGHER DIMENSIONS

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Let \( \Omega \) be a bounded region in \( \mathbb{R}^n \) and let \( \mathcal{P} = \{P_i\}_{i=1}^m \) be a partition of \( \Omega \) into a finite number of closed subsets having piecewise \( C^2 \) boundaries of finite \((n-1)\)-dimensional measure. Let \( \tau: \Omega \to \Omega \) be an expanding transformation on \( \mathcal{P} \) where, \( \tau_i = \tau|_{P_i} \) and \( \tau_i \in C^M, \ m \geq 2 \). We show that the \( \tau \)-invariant density \( h \in C^{M-2}\).

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1. Introduction

There has been a recent surge of interest in the study of existence and properties of absolutely continuous invariant measures (acim) of higher dimensional transformations. Let \( \Omega \) be a bounded region in \( \mathbb{R}^n \) and let \( \mathcal{P} = \{P_i\}_{i=1}^m \) be a partition of \( \Omega \) into a finite number of subsets having piecewise \( C^2 \) boundaries of finite \((n-1)\)-dimensional measure. Let \( \tau: \Omega \to \Omega \) be piecewise \( C^2 \) on \( \mathcal{P} \) where \( \tau_i = \tau|_{P_i} \) is a \( C^2 \) diffeomorphism onto its image and expanding in the sense that there exists \( \alpha > 1 \) such that for any \( i = 1, 2, \ldots, m, \| D\tau_i^{-1} \|_E < \alpha^{-1} \), where \( D\tau_i^{-1} \) is the derivative matrix of \( \tau_i^{-1} \) and \( \| \cdot \|_E \) is the Euclidean matrix norm. Then, under general conditions [1], it has been shown that \( \tau \) has an acim, which is a generalization of the results proved in [11, 12, 5] and [8]. We are then interested in properties associated with the acim. The
properties of interest include: number of ergodic acim [9, 2], uniqueness [4, 21], stability [13, 6] and the smoothness of their invariant densities.

It is important to know which properties the invariant density, if one exists, inherits from its underlying transformation. For Lasota-Yorke maps [15] on [0,1], Szewc [22] proved that the densities of invariant measures for Lasota-Yorke maps of class \( C^M \) are of class \( C^{M-1} \). It should be noted, however, that these smoothness properties are assumed to hold only piecewise - that is, relative to a partition of \( \Omega \). The smoothness of the invariant density in Szewc's result is actually piecewise-smoothness relative to another partition that is obtained from the given one through refinement with all of its forward images. Thus in most cases (with the exception of some simple classes of maps, such as Markov maps) the underlying partition for the piecewise smoothness of the invariant density has (possibly infinitely) many more elements than the original one.

In this paper, we investigate the smoothness of invariant densities of acim in higher dimensions for a subset of Lasota-Yorke maps. We prove that if a transformation is expanding in the sense of the maps considered by Mané [16] and of class \( C^M \), then its invariant density is of class \( C^{M-2} \). It should be noted, however, that this somewhat weaker smoothness result is valid on the original partition. We conjecture that this is the sharpest result possible for the given partition. In dimension one, the smoothness of the invariant density of an acim for transformations on an interval considered by Rényi [18] was established by Halfant [10]. An alternate proof of this has been presented in [17]. Some applications of expanding maps in a number theoretical context can be found in [20].

2. Definitions and Conditions

We denote by \( \lambda \), the Lebesgue measure on \( \mathbb{R}^n \). For a \( n \times n \times \ldots \times n \) (\( k \)-times) array \( M_k \) we define its norm by \( \| M_k \| = \max_{N_k} |(m)_{i_1 \ldots i_k}| \), where

\[
N_k = \{i_1i_2 \ldots i_k; 1 \leq i_j \leq n \text{ for } 1 \leq j \leq k \}.
\]

For a real-valued function \( f: \mathbb{R}^n \to \mathbb{R} \) we denote by \( Df \) its derivative, and by \( D^{(M)}f \) the \( M \)-th derivative of \( f \). If \( f(x) = f(x_1, x_2, \ldots, x_n) \) then \( (Df)_x \) is a linear map:

\[
(\mathbb{R}^n \to \mathbb{R} \text{ and } (Df)_x(v_1, v_2, \ldots, v_n) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}.
\]

Let \( \mathcal{P} = \{P_i\}_{i=1}^{m} \) be a partition of \( \Omega \) into a finite number of subsets having piecewise \( C^2 \) boundaries of finite \((n-1)\)-dimensional measure. Henceforth we work mainly with the open domain \( \Omega_0 = \Omega \setminus \bigcup_{i \in J_0} \partial P_i \). We use the notation \( \tau_t^{-k} \) to mean a specific composition of inverses of the form \( \tau_t^{-1} \circ \tau_{t-1}^{-1} \circ \ldots \circ \tau_{t-1}^{-1} \) where each \( i_j \in \{1, 2, \ldots, m\} \) for \( 1 \leq j \leq k \), and denote by \( J_k \) the set of subscripts \( t \) for which \( \tau_t^{-k} \) is defined (we use letters \( t,u, \ldots \) as subscript here to distinguish between \( \tau_t = \tau \mid P_t \) and the composition of inverses of such maps as described.) Thus for any open set \( A \), each \( \tau_t^{-k} \) is a \( C^M \) diffeomorphism of \( A \cap \tau_{i_1}^{-1}(P_{i_1}) \) onto its image. Of course, \( \tau_t^{-k}(A) = \bigcup_{t \in J_k} \tau_t^{-k}(A) \).

**Definition 1:** For an invariant measure \( \mu \), absolutely continuous with respect to \( \lambda \), the invariant density \( h \) is given by \( \mu = \int h \, d\lambda \).
Mané [16] defined a class of almost Markov expanding maps satisfying five conditions.

**Definition 2:** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $\Omega$ is a separable metric space and $\mathcal{F}$ its Borel $\sigma$-algebra. We say a map $\tau: \Omega \rightarrow \Omega$ is Mané-expanding if there exists a sequence of partitions $(\mathcal{P}_i)_{i \geq 0}$ of open sets such that:

(a) $\bigcup P \in \mathcal{P}_0 = \Omega \pmod{0}$.

(b) For every $k \geq 0$ and $P \in \mathcal{P}_{k+1}$, $\tau(P)$ is a union (mod $0$) of atoms of $\mathcal{P}_k$, and $\tau | P$ is injective.

(c) There exists $0 < \eta < 1$ and $K > 0$ such that,

$$\| \tau_t^{-k}(x) - \tau_t^{-k}(y) \|_E \leq K \eta^k \| x - y \|_E$$

for every $k \geq 0$, and for every $x, y$ in the domain of $\tau_t^{-k}$.

(d) There exists $l > 0$ such that, for every pair of atoms $P, Q \in \mathcal{P}_0$, we have $\mu(\tau^{-l}(P) \cap Q) \neq 0$.

(e) There exists $C > 0$ such that, for every $k \geq 0$ and $0 < \zeta \leq 1$ whenever $x, y$ are contained in the same atom of $\mathcal{P}_k$ we have

$$\left| \frac{\mathcal{F}_\tau(y)}{\mathcal{F}_\tau(x)} - 1 \right| \leq C \| \tau(x) - \tau(y) \|_{\mathcal{F}_\tau}.$$

($\mathcal{F}_\tau$ is the absolute value of the determinant of the Jacobian matrix of $\tau$).

Mané proved that these conditions are sufficient for the existence of an acim (see Theorem 1, Section 3).

**Remarks:** Such maps need not be Markov, but all Markov maps satisfy condition (b) (see [16, pp. 170-171]). The class of maps considered by Mané is, by condition (e), $C^1 + \xi$. This class contains $C^M; M \geq 2$.

For our considerations (assuming that an acim exists) only one of these conditions is needed:

**Expanding condition:** There exists $0 < \eta < 1$ and $K > 0$ such that,

$$\| \tau_t^{-k}(x) - \tau_t^{-k}(y) \|_E \leq K \eta^k \| x - y \|_E$$

for every $k \geq 0$, for every $t$ (corresponding to the maps $\tau_t^{-k}$ as described above) and for every $x, y$ in the domain of $\tau_t^{-k}$.

We will also need Mané’s condition (e), but, as remarked above, this condition is satisfied whenever $\tau$ is $C^M$, $M \geq 2$. The expanding condition implies that for a given $\varepsilon > 0$, for $N$ large enough and for any $u \in \mathcal{F}_N$ and any $x, y$ in the domain of $\tau_u^{-N}$ we have:

$$\| \tau_u^{-N}(x) - \tau_u^{-N}(y) \|_E / \| x - y \|_E < \varepsilon < 1.$$  \(1\)

Set $\tau_u^{-N} = (\phi_1, \phi_2, \ldots, \phi_n)$. Thus for a fixed $x$, if $y$ approaches $x$ in the direction of any of the $n$ coordinate axis, it follows that $\left| \frac{\partial \phi_j(x)}{\partial x_k} \right| < \varepsilon < 1$ where $1 \leq j, k \leq n$.

Thus there exists an $N$ such that for all $u \in \mathcal{F}_N$ we have

$$\| D\tau_u^{-N}(x) \| < \varepsilon < 1 \text{ for all } x \text{ in the domain of } \tau_u^{-N}. \quad (2)$$
For most of the results that follow we need only expanding in the sense of (2). In other words, if we assume that an acim exists, the results of this paper follow from the weaker expanding condition (2) and the distortion condition (see Lemma 1).

Definition 3: We define the measures \( \lambda_k \) by \( \lambda_k(A) = \lambda(\tau^{-k}(A)) \).

Definition 4: The Perron-Frobenius operator \( P_\tau : \mathcal{L}^1(\Omega) \to \mathcal{L}^1(\Omega) \) is
\[
P_\tau f(x) = \sum_{i=1}^{m} \frac{f(\tau^{-1}(x))}{\mathcal{F}_\tau(\tau^{-1}(x))}, \quad f \in \mathcal{L}^1(\Omega) \text{ and } m = \#(\mathcal{F}).
\]

Definition 5: We define iterated densities \( S_k(x) \) for almost all \( x \in \Omega \) by the finite sum \( S_k(x) = \sum_{t \in \mathcal{T}_k} S(\tau_t^{-k}) \). We note that, \( \int_A P_\tau^k 1 d\lambda = \int_{\tau^{-k}(A)} 1 d\lambda = \int_A d\lambda_k, \) and therefore \( S_k = \frac{d\lambda_k}{d\lambda} \).

3. Main Results

The following result was proved in [16, Chapter III, Theorem 1.3].

Theorem 1: (Mané). If \( \tau \) is Mané-expanding then it admits an invariant probability measure \( \mu \), absolutely continuous with respect to Lebesgue measure, and \( \lim_{k \to \infty} \lambda_k(A) = \mu(A) \) for all \( A \in \mathcal{B} \), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra of \( \Omega \).

The next result does not require the existence of an acim.

Theorem 2: Let \( \tau: \Omega \to \Omega \) satisfy the expanding condition and \( \tau \in C^M, \ M \geq 2 \). Then the sequence \( \{ \sup_{x \in \Omega} \| D^{(M-1)} S_k(x) \| \} \) is uniformly bounded.

Our final result, Theorem 3, states that if an acim exists, and if \( \tau \) is of class \( C^M \) and satisfies our expanding condition, then the density function must be of class \( C^{M-2} \). In this context, the existence of the acim is a separate problem, and, as stated in the introduction, there are many existence proofs in the literature. Of course, they all place some additional condition(s) on \( \tau \). For example, \( \tau \) could satisfy all of Mané’s conditions, and then Theorem 1 applies.

Theorem 3: Let \( \tau: \Omega \to \Omega \) admit an absolutely continuous invariant measure and satisfy the expanding condition, and let \( \tau \in C^M, \ M \geq 2 \). Then the invariant density \( h(x) \in C^{M-2} \) for all \( x \in \Omega_0 \).

4. Proofs and Lemmas

The following lemma establishes what is historically called the distortion condition.

Lemma 1: There exists a constant \( B^{(0)} \) such that for any \( k \geq 1 \) and all \( t \in \mathcal{T}_k \), we have
\[
\sup_{x \in \Omega} \frac{\mathcal{T}_{\tau_t^{-k}}(x)}{\inf_{x \in \Omega} \mathcal{T}_{\tau_t^{-k}}(x)} \leq B^{(0)}.
\]

Proof: We adapt the proof of [16] for dimension 1. From the definition of \( \mathcal{T}_\tau \) we have
\[
\frac{\mathcal{T}_{\tau_t^{-k}}(y)}{\mathcal{T}_{\tau_t^{-k}}(x)} = \frac{\mathcal{T}_{\tau_t^{-k}}(y)}{\mathcal{T}_{\tau_t^{-k}}(x)} = \prod_{j=0}^{k-1} \frac{\mathcal{T}_{\tau_t^{-j}}(y)}{\mathcal{T}_{\tau_t^{-j}}(x)}.
\]
\[ \leq \prod_{j=0}^{k-1} (1 + C \| \tau^{-j}w(x) - \tau^{-j}w(y) \| \xi), \]

where \( v \in \mathcal{J}_{j+1} \) (respectively \( w \in \mathcal{J}_j \)) corresponds to \( \tau^{k-(j+1)} \circ \tau^{-k} \) (respectively \( \tau^{k-j} \circ \tau^{-k} \)). Thus

\[
\frac{\mathcal{F}(\tau^{-k}(x))}{\mathcal{F}(\tau^{-k}(y))} \leq \prod_{j=0}^{k-1} (1 + CK^j \eta^j \| x - y \| \xi) \leq \prod_{j=0}^{\infty} (1 + CK^j \eta^j (\text{diam} \Omega)^j).
\]

This converges if and only if the series \( \sum_{j=0}^{\infty} \eta^j \) converges. Thus since \( \eta < 1 \),

\[
\sup_{x \in \Omega_0} \frac{\mathcal{F}(\tau^{-k}(x))}{\inf_{x \in \Omega_0} \mathcal{F}(\tau^{-k}(x))} \leq B^{(0)}.
\]

**Lemma 2:** \( S_k(x) \leq B^{(0)} \) for every \( x \in \Omega_0 \) and \( k \geq 1 \) (where \( B^{(0)} \) is the bound in Lemma 1).

**Proof:** Starting with Definition 5, the following relations are valid:

\[
\inf_{x \in \Omega_0} S_k(x) = \sup_{x \in \Omega_0} \sum_{t \in \mathcal{J}_k} \frac{\mathcal{F}(\tau^{-k}(x))}{\inf_{x \in \Omega_0} \mathcal{F}(\tau^{-k}(x))} \leq \sum_{t \in \mathcal{J}_k} \sup_{x \in \Omega_0} \frac{\mathcal{F}(\tau^{-k}(x))}{\inf_{x \in \Omega_0} \mathcal{F}(\tau^{-k}(x))} \leq B^{(0)}.
\]

Since \( \int \Omega S_k d\lambda = 1 \), it follows that \( \frac{1}{B^{(0)}} \leq \inf_{x \in \Omega_0} S_k(x) \leq \sup_{x \in \Omega_0} S_k(x) \leq B^{(0)}. \)

The following lemma was proved in [7] in dimension 1.

**Lemma 3:** Let \( F(x) = f(\phi(x)) \mathcal{F}_{\phi}(x) \) for all \( x \in \Omega \). Then

\[
D^M F = D^M f(\phi(D\phi)^M \mathcal{F}_{\phi} + \sum_{j=0}^{M-1} \left( D^j f(\phi) \left( \sum_{i=0}^{M} \frac{D^j \mathcal{F}_{\phi} P_{i,j,M}(D\phi, \ldots, D^M \phi)}{} \right) \right),
\]

where \( P_{i,j,M}(D\phi, \ldots, D^M \phi) \) is a polynomial.

**Proof:** Straightforward proof by induction.

We now prove that the sequence \{ \( \| D^M S_k(x) \| \) \} is uniformly bounded, if \( \tau \) is piecewise \( C^M \). This will guarantee that for every \( 0 \leq j \leq M-2 \), the sequence \{ \( \| D^j S_k(x) \| \) \} is also uniformly bounded and equicontinuous.

**Proof of Theorem 2:** We prove the theorem by induction. First we note that
Let $M = 2$. By differentiation we obtain

$$DS_K + N(x) = \sum_{u \in \mathcal{J}_N} \{DS_K(\tau_u - N(x))D\tau_u^{-N(x)} \nabla_{\tau_u - N(x)} S_K(\tau_u - N(x)) + S_K(\tau_u - N(x))D\nabla_{\tau_u - N(x)} S_K(\tau_u - N(x))\}.$$  

We take $N$ so that (2) is satisfied (for $\varepsilon = \frac{1}{B(0)}$). We put

$$\gamma = \sup_{x \in \Omega_0} \|D\tau_u^{-N(x)}\| \leq \frac{1}{B(0)} < 1.$$ 

Since $\tau$ is piecewise $C^2$ on $\Omega$, we also have

$$\beta_1^{(2)} = \sup_{x \in \Omega_0} \sum_{u \in \mathcal{J}_N} \|D\nabla_{\tau_u - N(x)} S_K(\tau_u - N(x))\| < \infty \text{ and } B_k \sup_{x \in \Omega_0} \|DS_k(x)\| < \infty.$$ 

Using Lemma 2 and Definition 5, we obtain $B_K + N \leq B_K^2 B(0) + B(0) \beta_1^{(2)}$. This implies that the sequence $\{B_K + iN\}_{i=1}^{\infty}$, is uniformly bounded by some number $B_K$ and hence the entire sequence $\{B_k\}$ is bounded by $B(1) = \max_0 \leq i \leq N - 1 \{B_i\}$. Thus $\{\sup_{x \in \Omega_0} \|DS_k(x)\|\}$ is uniformly bounded by $B(1)$.

Now we assume the theorem is true for $M$ and prove it for $M + 1$ (i.e., we assume that $\tau \in C^{M+1}$ and prove that $\{\sup_{x \in \Omega_0} D^{(M)} S_k(x)\}$ is uniformly bounded).

By Lemma 3 we have

$$D^{(M)} S_K + N(x) = \sum_{u \in \mathcal{J}_N} \left\{D^{(M)} S_K(\tau_u - N(x))(D\tau_u^{-N(x)})^M \nabla_{\tau_u - N(x)} + \sum_{j=0}^{M-1} \left(D^{(j)} S_K(\tau_u - N(x)) \sum_{i=0}^{M} \left(D\nabla_{\tau_u - N(x)} P_{i,j,M}(D\tau_u^{-N(x)},\ldots,D^M\tau_u^{-N(x)})\right)\right)\right\}.$$ 

Now by induction we have, for $j = 0, 1, \ldots, M - 1$, constants $B^{(j)}$ which are bounds for the sequences $\{\sup_{x \in \Omega_0} D^{(j)} S_k(x)\}$ respectively.

Since $\tau$ is piecewise $C^{M+1}$ on $\Omega$, for each $j$ there exists a constant $\beta_j^{(M)}$ such that

$$\beta_j^{(M)} = \sum_{u \in \mathcal{J}_N} \sum_{i=0}^{M} \left(D\nabla_{\tau_u - N(x)} P_{i,j,M}(D\tau_u^{-N(x)},\ldots,D^M\tau_u^{-N(x)})\right) < \infty.$$ 

Let $B_k^{(M)} = \sup_{x \in \Omega_0} \|D^{(M)} S_k\| \infty (x) < \infty$. Then

$$B_K^{(M)} + N \leq B_K^{(M)} \gamma^M B(0) + \sum_{j=0}^{M-1} \beta_j^{(M)} \beta_j^{(M)}.$$ 

Since $\gamma^M B(0) < 1$, the sequence $\{B_K^{(M)} + iN\}_{i=1}^{\infty}$ is uniformly bounded by some
Lemma 4: Let \( f: \mathbb{R}^n \to \mathbb{R} \) and \( \{S_k\} \) be a sequence of functions \( S_k: \mathbb{R}^n \to \mathbb{R} \) such that \( S_k \to f \). If \( D S_k \to g \) (uniformly) then \( D f = g \).

Proof: This is a standard result, e.g., Theorem 7.17 of [19].

Proof of Theorem 3: Using Theorem 2, we conclude that the sequence \( \{D(K)S_k\} \) is uniformly bounded and equicontinuous for \( 0 \leq K \leq M - 2 \) on \( \Omega_0 \). By the Ascoli-Arzela Theorem, there is a subsequence \( \{D(K)S_k\} \) with a continuous limit, \( f_K(x) \).

In particular, we have \( \lim_{l \to \infty} S_{k_l}(x) = f_0(x) = f(x) \) \( \forall x \in \Omega_0 \). From Lemma 4, it follows that \( f_K(x) = D(K)f(x) \) for \( K = 0, 1, 2, \ldots, M - 2 \) and for all \( x \in \Omega_0 \). To complete the proof, we note that since \( \lim_{l \to \infty} \int S_{k_l} d\lambda = \lim_{l \to \infty} \lambda_{k_l} = \int f d\lambda \), and since we know \( \{\lambda_{k_l}\} \) converges (Theorem 1) it follows that \( \lim_{k \to \infty} \lambda_{k}(A) = \int_A f d\lambda = \mu(A) \).

This means \( f \) is equal to the invariant density \( h \) a.e. Hence, \( h \) can be chosen to be \( M - 2 \) times differentiable and we have \( f^{(j)} = h^{(j)} \), for \( 1 \leq j \leq M - 2 \).

Corollary 3.1: Let \( \tau: \Omega \to \Omega \) be expanding and \( \tau_1 \in C^M, M \geq 2 \). Then for \( 0 \leq j \leq M - 2 \), the sequence \( \{D(j)S_k\} \) converges uniformly to \( h^{(j)} \).

Proof: This follows immediately from Theorem 3 and the fact that \( \{D(j)S_{k_l}\} \) is uniformly bounded and equicontinuous.

5. Conclusions

Our results could be improved in two directions. One problem to consider is to establish the smoothness property of invariant densities for Lasota-Yorke maps in higher dimensions. Another problem would be to increase the degree of smoothness from \( C^{M-2} \) to \( C^{M-1} \), (but as noted in the introduction this would in general require a finer partition). In one dimension, this was proved by [22]. Furthermore, it seems possible to establish the existence of an acim for random maps (see [3] and [14]) composed of expanding transformations, and to derive smoothness properties of invariant densities based on the technique used in this paper.

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