CONVERGENCE OF AN ITERATION LEADING TO A SOLUTION OF A RANDOM OPERATOR EQUATION

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In the present paper, we define a random iteration scheme and consider its convergence to a solution of a random operator equation. There is also a related fixed point result.

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1. Introduction

In recent years, the study of different types of random equations have attracted much attention, some of which may be noted in [1, 5, 6] and [7]. In this paper, we discuss a random operator equation involving two operators in the context of Hilbert spaces. We have also a random fixed point result as a corollary. We also demonstrate our result for the corresponding deterministic case by an example.

Throughout this paper, \((\Omega, \Sigma)\) denotes a measurable space and \(H\) stands for a separable Hilbert space.

A function \(f: \Omega \to H\) is said to be measurable if \(f^{-1}(B) \in \Sigma\) for every Borel subset \(B\) of \(H\).

A function \(F: \Omega \times H \to H\) is said to be \(H\)-continuous, if \(F(t, \cdot):H \to H\) is continuous for all \(t \in \Omega\).

A function \(F: \Omega \times H \to H\) is said to be a random operator, if \(F(\cdot, x):\Omega \to H\) is measurable for every \(x \in H\).

A measurable function \(g: \Omega \to H\) is said to be a random fixed point of the random operator \(F: \Omega \times H \to H\), if \(F(t, g(t)) = g(t)\) for all \(t \in \Omega\).

A measurable function \(g: \Omega \to H\) is said to be a solution of the random operator equation \(S(t, x(t)) = T(t, x(t))\), where \(S, T: \Omega \times H \to H\) are random operators, if \(S(t, g(t)) = T(t, g(t))\) for all \(t \in \Omega\).
Lemma 1.1: Let $H$ be a Hilbert space. Then for any $x,y,z \in H$ and any real $\lambda$, the following equality holds:

$$
\| (1 - \lambda)x + \lambda y - z \|^2 = \| x - z \|^2 + \lambda \| y - z \|^2 - \lambda(1 - \lambda) \| x - y \|^2.
$$

We define the random iteration scheme as follows:

**Definition 1.2:** Random iteration scheme. Let $S,T: \Omega \times H \rightarrow H$ be two random operators defined on a Hilbert space $H$. Let $g_0: \Omega \rightarrow H$ be any measurable function. Define the following sequence of functions $\{g_n\}$

$$
{g_{n+1}(t) = (1 - \alpha_n)g_n(t) + \alpha_nh_n(t),}
$$

where

$$
h_n(t) = (1 - \beta_n)S(t, g_n(t)) + \beta_nT(t, g_n(t)),
$$

$$
0 < \alpha_n, \beta_n < 1 \text{ for all } n = 0, 1, 2, \ldots,
$$

and

$$
\lim_{n \to \infty} \beta_n = M < 1,
$$

and

$$
\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty.
$$

The construction of the iteration scheme is based on the same idea as that of Ishikawa’s random iteration scheme [2]. But the present iteration is not a modification or generalization of that iteration.

A function $T: H \rightarrow H$ is said to satisfy Tricomi’s condition if

$$
Tp = p \text{ implies } \| Tx - p \| \leq \| x - p \|. 
$$

We define generalized Tricomi’s condition for two operators in the following way.

**Definition 1.3:** Generalized Tricomi’s Condition. Two functions $S,T: H \rightarrow H$ are said to satisfy generalized Tricomi’s condition if

$$
Sp = Tp \text{ implies } \| Sx - p \| \leq \| x - p \|
$$

and

$$
\| Tx - p \| \leq \| x - p \|.
$$

2. Main Results

**Theorem 2.1:** Let $S,T: \Omega \times H \rightarrow H$, where $H$ is a separable Hilbert space, be two random operators such that
(a) $S$ and $T$ are $H$-continuous, and
(b) there exists $f: \Omega \to H$ (not necessarily measurable) such that

\[(1 - \lambda) \| S(t,x) - f(t) \|^2 + \lambda \| T(t,x) - f(t) \|^2 \leq \| x - f(t) \|^2 \quad (2.1)\]

for all $t \in \Omega$, $x \in H$ and $0 < \lambda < 1$.

Then the random iteration scheme (Definition 1.2), if convergent, converges to a solution of the random operator equation

\[S(t,x(t)) = T(t,x(t)). \quad (2.2)\]

**Proof:** For any $t \in \Omega$,

\[
\begin{align*}
\| g_{n+1}(t) - f(t) \|^2 &= \| (1 - \alpha_n)g_n(t) + \alpha_n h_n(t) - f(t) \|^2 \\
&= (1 - \alpha_n) \| g_n(t) - f(t) \|^2 + \alpha_n \| h_n(t) - f(t) \|^2 \\
&\quad - \alpha_n(1 - \alpha_n) \| g_n(t) - h_n(t) \|^2 \\
&\leq (1 - \alpha_n) \| g_n(t) - f(t) \|^2 \\
&\quad + \alpha_n \| (1 - \beta_n)S(t,g_n(t)) + \beta_n T(t,g_n(t)) - f(t) \|^2 \\
&= (1 - \alpha_n) \| g_n(t) - f(t) \|^2 + \alpha_n \{ (1 - \beta_n) \| S(t,g_n(t)) - f(t) \|^2 \\
&\quad + \beta_n \| T(t,g_n(t)) - f(t) \|^2 \} \\
&\quad + \alpha_n \{ (1 - \beta_n) \| S(t,g_n(t)) - f(t) \|^2 + \beta_n \| T(t,g_n(t)) - f(t) \|^2 \} \\
\end{align*}
\]

or,

\[
\begin{align*}
\alpha_n \beta_n(1 - \beta_n) \| S(t,g_n(t)) - T(t,g_n(t)) \|^2 \\
&\leq (1 - \alpha_n) \| g_n(t) - f(t) \|^2 - \| g_{n+1}(t) - f(t) \|^2 \\
&\quad + \alpha_n \{ (1 - \beta_n) \| S(t,g_n(t)) - f(t) \|^2 + \beta_n \| T(t,g_n(t)) - f(t) \|^2 \} \\
&\quad + \alpha_n \{ (1 - \beta_n) \| S(t,g_n(t)) - f(t) \|^2 + \beta_n \| T(t,g_n(t)) - f(t) \|^2 \} \quad (2.3) \\
\end{align*}
\]

or,

\[
\begin{align*}
\alpha_n \beta_n(1 - \beta_n) \| S(t,g_n(t)) - T(t,g_n(t)) \|^2 \\
&\leq (1 - \alpha_n) \| g_n(t) - f(t) \|^2 - \| g_{n+1}(t) - f(t) \|^2 \quad \text{for all } t \in \Omega \quad (by \ (2.1)) \\
\end{align*}
\]

or,

\[
\begin{align*}
\alpha_n \beta_n(1 - \beta_n) \| S(t,g_n(t)) - T(t,g_n(t)) \|^2 \\
&\leq \| g_n(t) - f(t) \|^2 + \alpha_n \| g_n(t) - f(t) \|^2 \quad \text{for all } t \in \Omega. \quad (2.4) \\
\end{align*}
\]
Summing up the inequalities in (2.4) over \(n\), we obtain for all \(t \in \Omega\),
\[
\sum_{n=0}^{\infty} \alpha_n \beta_n (1 - \beta_n) \| S(t, g_n(t)) - T(g, g_n(t)) \|^2 \leq \| g_0(t) - f(t) \|^2 < \infty. \tag{2.5}
\]

Let \(M < M' < 1\). Then, by (1.5), there exists a positive integer \(m_0\) such that \(\beta_m < M'\), that is \(1 - \beta_m > 1 - M'\) for all \(m > m_0\).

This shows that
\[
\sum_{m=m_0}^{\infty} \alpha_m \beta_m (1 - \beta_m) \geq (1 - M') \sum_{m=m_0}^{\infty} \alpha_m \beta_m = \infty. \tag{2.6}
\]

(2.5) and (2.6) imply that, for all \(t \in \Omega\),
\[
\lim_{n \to \infty} \| S(t, g_n(t)) - T(t, g_n(t)) \|^2 = 0. \tag{2.7}
\]

Let
\[
g_n(t) \to g(t) \text{ as } n \to \infty. \tag{2.8}
\]

Since \(g_0\) is measurable and \(H\) is separable, according to Himmelberg [3], \(g_n\)'s are measurable and, therefore, \(g: \Omega \to H\) is measurable.

Again, \(S\) and \(T\) are \(H\)-continuous, which shows that \(\lim_{n \to \infty} S(t, g_n(t)) = S(t, g(t))\) and
\[
\lim_{n \to \infty} T(t, g_n(t)) = T(t, g(t)) \text{ for all } t \in \Omega.
\]

By (2.7) and (2.8), we have for all \(t \in \Omega\),
\[
S(t, g(t)) = T(t, g(t)), \text{ where } g: \Omega \to H \text{ is a measurable function.} \tag{2.9}
\]

This shows that the random iteration scheme if convergent, converges to a solution of (2.2).

**Corollary 2.2:** Let \(H\) be a separable Hilbert space and \(S, T: \Omega \times H \to H\) be two random operators such that
(a) \(S, T\) are \(H\)-continuous, and
(b) there exists \(f: \Omega \to H\) (not necessarily measurable) such that
\[
\| S(t, x) - f(t) \| \leq \| x - f(t) \| \tag{2.10}
\]
and
\[
\| T(t, x) - f(t) \| \leq \| x - f(t) \|. \tag{2.11}
\]

Then the random iteration scheme if convergent, converges to a solution of \(S(t, x(t)) = T(t, x(t))\).

**Proof:** It is easily seen that (2.10) and (2.11) imply (2.1). The corollary then follows by Theorem 2.1.

Setting \(S\) as the identify random operator, that is, \(S(t, x) = x\) for all \(t \in \Omega\) and
Corollary 2.3: Let $H$ be a separable Hilbert space and $T$ be a random operator which is $H$-continuous. Assume that there exists $f: \Omega \to H$ (not necessarily measurable) such that for all $t \in \Omega$

$$\| T(t, x) - f(t) \| \leq \| x - f(t) \|. \quad (2.12)$$

Then the sequence of functions $\{g_n\}$, where $g_0: \Omega \to H$, is measurable and

$$g_{n+1}(t) = (1 - \alpha_n)g_n(t) + \alpha_n((1 - \beta_n)g_n(t) + \beta_nT(t, g_n(t))), \quad n = 0, 1, 2, \ldots \quad (2.13)$$

for all $t \in \Omega$, where $0 < \alpha_n, \beta_n < 1$, $\prod_{n=1}^{\infty} \beta_n < 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ if convergent, converges to a random fixed point of $T$.

Corollary 2.4: Let $S, T: H \to H$ be two operators such that the following holds: there exists $z \in H$ such that

$$(1 - \lambda)\| Sx - z \|^2 + \lambda\| Tx - z \|^2 \leq \| x - z \|^2, \quad (2.14)$$

for all $x \in H$ and $0 < \lambda < 1$.

Then the sequence $\{X_n\}$, obtained by the iteration

$$x_0 \in H, \quad (2.15)$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n((1 - \beta_n)Sx_n + \beta_nTx_n), \quad (2.16)$$

where

$$0 < \alpha_n, \beta_n < 1 \quad (2.17)$$

$$\lim_{n \to \infty} \beta_n < 1 \quad (2.18)$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \quad (2.19)$$

if convergent, converges to a solution of $Sx = Tx$.

The proof trivially follows from Theorem 2.1. It may be noted that the separability of $H$ was required to ensure that $g_n$'s are measurable. In the statement of the corollary, $H$ need not be separable.

Theorem 2.5: Let $C$ be a convex and compact subset of a separable Hilbert space $H$ and $S, T: \Omega \times H \to \Omega \times H$ be two random operators such that the following conditions are satisfied:

(a) $S, T$ are $H$-continuous,

(b) there exists $f: \Omega \to H$ (not necessarily measurable) such that

$$(1 - \lambda)\| S(t, x) - f(t) \|^2 + \lambda\| T(t, x) - f(t) \|^2 \leq \| x - f(t) \|^2 \quad (2.20)$$

for all $t \in \Omega$, $x \in H$ and $0 < \lambda < 1$, and
(c) $S(t, \cdot), T(t, \cdot): C \rightarrow C$ satisfy Generalized Tricomi’s condition for all $t \in \Omega$. Then for any measurable function $g_0: \Omega \rightarrow C$, the sequence of functions $\{g_n\}$ constructed by the random iteration scheme ((1.2)-(1.6)) actually converges to a solution of the random operator equation $S(t, x(t)) = T(t, x(t))$.

**Proof:** By the construction of $\{g_n\}$ it is seen that $g_n$’s are measurable functions from $\Omega$ to $C$ for all $n = 0, 1, 2, \ldots$. Proceeding exactly in the same way as in Theorem 2.1, we have as in (2.7) that

$$\lim_{n \to \infty} || S(t, g_n(t)) - T(t, g_n(t)) ||^2 = 0.$$ 

Therefore, for a fixed $t \in \Omega$, there exists a subsequence

$$\{g_{n_1}(t)\} \subset \{g_n(t)\} \text{ such that } \lim_{i \to \infty} || S(t, g_{n_1}(t)) - T(t, g_{n_1}(t)) || = 0. \quad (2.21)$$

Again, $C$ is compact, therefore, there exists $\{g_{n_1}(t)\} \subset \{g_{n_1}(t)\}$ such that $\{g_{n_1}(t)\}$ is convergent.

Let

$$\lim_{k \to \infty} g_{n_1_k}(t) = g(t) \text{ for } t \in \Omega. \quad (2.22)$$

Since $S$ and $T$ are $H$-continuous random operators, from (2.21), we have for any $t \in \Omega$,

$$S(t, g(t)) = T(t, g(t)). \quad (2.23)$$

For any $t \in \Omega$,

$$|| g_{n+1}(t) - g(t) ||^2 = || (1 - \alpha_n)g_n(t) + \alpha_n h_n(t) - g(t) ||^2$$

$$= (1 - \alpha_n) || g_n(t) - g(t) ||^2 + \alpha_n || h_n(t) - g(t) ||^2$$

$$- (1 - \alpha_n) \alpha_n || g_n(t) - h_n(t) ||^2$$

$$\leq (1 - \alpha_n) || g_n(t) - g(t) ||^2 + \alpha_n || (1 - \beta_n) S(t, g_n(t)) + \beta_n T(t, g_n(t)) - g(t) ||^2$$

(by (1.3) and (1.4))

$$= (1 - \alpha_n) || g_n(t) - g(t) ||^2 + \alpha_n ((1 - \beta_n) || S(t, g_n(t)) - g(t) ||^2$$

$$+ \beta_n || T(t, g_n(t)) - g(t) ||^2) - \alpha_n \beta_n (1 - \beta_n) || S(t, g_n(t)) - T(t, g_n(t)) ||^2$$

(by (1.1))

$$\leq (1 - \alpha_n) || g_n(t) - g(t) ||^2 + \alpha_n || g_n(t) - g(t) ||^2$$

(by (2.23) and Generalized Tricomi’s condition)

or
\[ ||g_{n+1}(t) - g(t)|| \leq ||g_n(t) - g(t)||. \]  
(2.24)

(2.22) and (2.24) together imply that

\[ g_n \to g \text{ as } n \to \infty. \]  
(2.25)

Since \( H \) is separable, \( g_n \)'s are measurable [3], and, hence \( g \) is also measurable. From (2.23), \( g \) is a random solution of \( S(t,x(t)) = T(t,x(t)) \). This completes the proof. \( \square \)

We have the following obvious corollary.

**Corollary 2.6:** Let \( S, T: H \to C \), where \( C \) is a compact and convex subset of a Hilbert space \( H \) are such that the following are satisfied:

(a) \( S, T \) are continuous,
(b) there exists \( z \in H \) such that

\[ (1 - \lambda) ||Sx - z||^2 + \lambda ||Tx - z||^2 \leq ||x - z||^2 \]  
(2.26)

for all \( x \in H \) and \( 0 < \lambda < 1 \), and
(c) \( S, T \) satisfy Generalized Tricomi's condition.

Then the sequence, defined as \( x_0 \in C \),

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n((1 - \beta_n)Sx_n + \beta_nTx_n), \quad n = 0, 1, 2, \ldots \]  
(2.27)

where

\[ 0 < \alpha_n, \beta_n < 1, \]  
(2.28)

\[ \lim_{n \to \infty} \beta_n < 1, \]  
(2.29)

and

\[ \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty, \]  
(2.30)

converges to a solution of the equation \( Sx = Tx \).

**Example:** Let \( C = [0, 1] \), \( S, T: \mathbb{R} \to [0, 1] \) be defined as

\[ Sx = x^2/2 \text{ if } x \in [0, 1] \]

\[ = 1/2 \text{ if } x > 1 \]

\[ = 0 \text{ if } x < 0 \]

and

\[ Tx = x^2/4 \text{ if } x \in [0, 1] \]

\[ = 1/4 \text{ if } x > 1 \]

\[ = 0 \text{ if } x < 1. \]
With the choice of $z = 0$, the conditions of Corollary 2.6 are seen to be satisfied. Thus Corollary 2.6 applies to this example.

References