We show that $\mathcal{F}_t$-adapted, set-valued stochastic processes satisfying mild continuity conditions admit, $\mathcal{F}_t$-adapted, stochastically continuous selections.

**Key words:** Set-valued Stochastic Process, Conditional Expectation, Martingale, Measurable and Continuous Selections.

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1. Introduction

In this paper we prove several theorems on the existence of $\mathcal{F}_t$-adapted, continuous selections for $\mathcal{F}_t$-adapted, set-valued stochastic processes, as well as a continuous time version of Hess' result on martingale selection [3]. Such results may be useful in the theory of the set-valued stochastic integral.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ (i.e., with a family of $\sigma$-fields $\mathcal{F}_t$), such that $0 \leq s \leq t$ implies that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. We assume that all $P$-null sets are in $\mathcal{F}_0$. Let $\mathcal{F}_t^- = \sigma(\bigcup_{s \geq t} \mathcal{F}_s)$ and $\mathcal{F}_t^+ = \bigcap_{s > t} \mathcal{F}_s$. Obviously, $\mathcal{F}_t^- \subseteq \mathcal{F}_t \subseteq \mathcal{F}_t^+$.

For a random variable $\varphi: \Omega \rightarrow \mathbb{R}^n$ such that $E(|\varphi|) = \int_{\Omega} |\varphi| \, dP < +\infty$, by $E(\varphi \mid \mathcal{F}_t)$ we denote the conditional expectation of $\varphi$, (i.e., an $\mathcal{F}_t$-measurable mapping) such that

$$\int_{\mathcal{A}} E(\varphi \mid \mathcal{F}_t) \, dP = \int_{\mathcal{A}} \varphi \, dP$$

for each $\mathcal{A} \in \mathcal{F}_t$.

We say that a set-valued mapping $\Phi: \Omega \rightarrow \mathbb{R}^n$ is a set-valued random variable iff $\Phi$ is $\mathcal{F}$-measurable (weakly measurable in the terminology of Himmelberg [5]), i.e.,

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\{\omega: \Phi(\omega) \cap U \neq \emptyset\} \in \mathcal{F} \text{ for each open set } U \subseteq \mathbb{R}^n. \text{ Equivalently, } \Phi \text{ is } \mathcal{F}\text{-measurable iff the real-valued function } d(z, \Phi): \Omega \to \mathbb{R}^n \text{ defined by}
\[
d(z, \Phi)(\omega) = d(z, \Phi(\omega)) = \inf_{\omega \in \Phi(\omega)} ||z - \omega||,
\]
where \(||w||\) is the Euclidean norm of \(w \in \mathbb{R}^n\), is a random variable. Clearly, for a mapping \(\varphi: \Omega \to \mathbb{R}^n\) identified with the set-valued mapping \(\Phi = \{\varphi\}\), this is equivalent to saying that \(\varphi\) is a random variable. Let \((F_t) = (F_t)_t \geq 0\) be a \textit{set-valued stochastic process} with closed values in \(\mathbb{R}^n\) (i.e., a family of \(\mathcal{F}\text{-measurable set-valued mappings } F_t: \Omega \to \mathbb{R}^n, t \geq 0, \text{ with closed values}\). We say that \((F_t)\) is \(\mathcal{F}_t\text{-adapted}\) iff \(F_t\) is \(\mathcal{F}_t\text{-measurable}\) for each \(t \geq 0\), and we denote an \(\mathcal{F}_t\text{-adapted process } (F_t)\) such that \(E(d(0, F_t)) < +\infty \text{ for each } t \geq 0, \text{ by } (F_t, \mathcal{F}_t)\). A \textit{selection} of the process \((F_t)\) is a single-valued stochastic process \((f_t)\) such that for every \(t \geq 0\), there holds \(f_t(\omega) \in F_t(\omega)\) for \(P\)-almost all \(\omega\). Additionally, if \((f_t)\) is \(\mathcal{F}_t\text{-adapted} and satisfies \(E(||f_t||) < +\infty \text{ for each } t \geq 0, \text{ we will denote the process by } (f_t, \mathcal{F}_t)\).

Let us mention that for the unique \(\sigma\)-field \(\mathcal{F}\), the result on convergence of measurable selections being extracted from the sequence of measurable set-valued mappings, that converge in the distribution, has been investigated by Salinetti and Wets [9, Theorem 5.1, Corollary 5.2]. On the other hand, Hess has proven the existence of martingale selections for discrete time, set-valued martingales and discussed the convergence of set-valued martingales.

### 3. Selection Theorem Results

Our first simple result concerns the case when almost all paths \(t \mapsto F_t(\omega)\) are continuous, and similar to the results of Salinetti and Wets, are based on the regularity of metric projections. For \(z \in \mathbb{R}^n\) and the closed, convex set \(A \subseteq \mathbb{R}^n\), we denote by \(\text{Pr}(z, A)\) the \textit{metric projection of } \(z\) \textit{onto } \(A\) \textit{with respect to Euclidean norm} (i.e., a unique element \(\text{Pr}(z, A) \in A\) such that \(||\text{Pr}(z, A) - z|| = d(z, A)\)). The \textit{Wijsman topology} for the family \(\text{CCI}(\mathbb{R}^n)\) of all nonempty, closed convex subsets of \(\mathbb{R}^n\), is the weakest topology such that for every \(y \in \mathbb{R}^n\), the function \(A \mapsto d(y, A)\) is continuous [10]. We will need the following lemma.

**Lemma 1:** The mapping \(A \mapsto \text{Pr}(z, A)\) of \(\text{CCI}(\mathbb{R}^n)\) into \(\mathbb{R}^n\) is continuous with respect to the Wijsman topology.

**Proof:** For \(A, A_0 \in \text{CCI}(\mathbb{R}^n)\) and \(z \in \mathbb{R}^n\), let us denote \(y_0 = \text{Pr}(y_0, A), y = \text{Pr}(z, A)\). Clearly,
\[
||y - y_0|| \leq ||y - \text{Pr}(y_0, A)|| + ||\text{Pr}(y_0, A) - y_0|| = ||y - \text{Pr}(y_0, A)|| + d(y_0, A).
\]

By the parallelogram equality, we have
\[
||y - \text{Pr}(y_0, A)||^2 = 2||y - z||^2 + 2||\text{Pr}(y_0, A) - z||^2 - 4\left(\frac{y + \text{Pr}(y_0, A)}{2} - z\right)^2 \leq 2||\text{Pr}(y_0, A) - z||^2 - 2d(z, A)^2.
\]

But
\[ \| \Pr(y_0, A) - z \| \leq \| \Pr(y_0, A) - y_0 \| + \| y_0 - z \| = d(y_0, A) + d(z, A_0). \]

Thus,
\[ \| y - \Pr(y_0, A) \|^2 \leq 2(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A)). \]

Consequently,
\[ \| y - y_0 \| \leq d(y_0 < A) + \sqrt{2}(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A)). \]

From this it follows immediately that \( A \rightarrow \Pr(z, A) \) is continuous with respect to the Wijsman topology.

**Theorem 1:** If the stochastic process \((F_t, \mathcal{F}_t)\) has closed convex values and for every \( z \in \mathbb{R}^n \), the functions \( t \rightarrow d(z, F_t)(\omega) \) is continuous for a.e. \( \omega \in \Omega \), then for any \( y \in \mathbb{R}^n \), the process \((f_t) \) defined by \( f_t(\omega) = \Pr(y, F_t(\omega)) \) is an \( \mathcal{F}_t \)-adapted selection of \( F \) such that \( t \rightarrow f_t(\omega) \) is continuous for \( P \)-a.e. \( \omega \in \Omega \).

**Proof:** By virtue of Lemma 1, from the assumption that the functions \( t \rightarrow d(z, F_t)(\omega) \), \( z \in \mathbb{R}^n \), and a.e. \( \omega \in \Omega \) are continuous, it follows that for every \( y \in \mathbb{R}^n \), a.e. \( \omega \in \Omega \), the mapping \( t \rightarrow \Pr(y, F_t(\omega)) \) is continuous. To see that \( f_t \) is \( \mathcal{F}_t \)-measurable note that
\[ \text{Graph } f_t = \{(\omega, z) : \| y - z \| - d(y, F_t(\omega)) < 0 \} \cap \text{Graph } F_t. \]

Hence, by virtue of [5, Theorem 3.5 and Corollary 6.3], \( f_t \) is \( \mathcal{F}_t \)-measurable.

In the following theorems we dispense completely with the upper semicontinuity assumption for the process \((F_t, \mathcal{F}_t)\). We do not adopt any lower semicontinuity assumption for the functions \( t \rightarrow d(y, F_t)(\omega) \); we assume only the stochastic upper semicontinuity of these functions, which means the stochastic lower semicontinuity of the process \((F_t, \mathcal{F}_t)\). We utilize a well-known theorem on measurable selections due to Kuratowski and Ryll-Nardzewski, as well as theorems on continuous selections of lower semicontinuous, set-valued mappings due to Michael [7] and to Antosiewicz, Cellina (see e.g., [1, Theorem 3]), respectively. We will need the following lemma.

**Lemma 2:** Assume that for the stochastic process \((F_t, \mathcal{F}_t)\), \( s \geq 0 \) and every \( z \in \mathbb{R}^n \), \( A \in \mathcal{F}_s \), the real-valued function \( t \rightarrow E(X_A d(z, F_t)) \) is right-hand (respectively: left-hand) usc at \( s \). Then for any \( \mathcal{F}_s \)-measurable random variable \( \varphi \) with \( E(\varphi) < +\infty \), the function \( t \rightarrow E(d(\varphi, F_t)) \) is right-hand (respectively: left-hand) usc at \( s \).

**Proof:** Let \( \epsilon > 0 \). By assuming that for any constant function, \( \varphi \equiv z \), we have \( E(d(\varphi, F_t)) < E(d(\varphi, F_s)) + \frac{\epsilon}{2} \) whenever \( t \in [s, s + \delta] \) (respectively, \( t \in (s - \delta, s] \)) for sufficiently small \( \delta \). For a step random variable \( \varphi = \sum_{i=1}^m z_i 1_{A_i} \), \( \varphi \in \mathcal{F}_s \), we have
\[ E(d(\varphi, F_t)) = \sum_{i=1}^m E(X_{A_i} d(z_i, F_t)) \leq \sum_{i=1}^m (E(X_{A_i} d(z_i, F_s)) + \frac{\epsilon}{2^i}) \leq E(d(\varphi, F_s)) + \epsilon, \]
whenever \( t \in [s, s + \delta] \) \((t \in (s - \delta, s]) \) for sufficiently small \( \delta \).
\( E( | \varphi - \varphi_n | ) \to 0 \). Then choose \( n \) such that \( E( | \varphi - \varphi_n | ) < \frac{\epsilon}{3} \) and let \( \delta > 0 \) be such that \( E(d(\varphi_n, F_t)) < E(d(\varphi_n, F_s)) + \frac{2\epsilon}{3} \) for \( t \in [s, s + \delta) \) \( (t \in (s - \delta, s]) \). Then,

\[
E(d(\varphi, F_t)) \leq E( | \varphi - \varphi_n | ) + E(d(\varphi_n, F_t)) < E(d(\varphi_n, F_s)) + \frac{2\epsilon}{3} < E(d(\varphi, F_s)) + \epsilon,
\]

whenever \( t \in [s, s + \delta) \) \( (t \in (s - \delta, s]) \).

**Theorem 2:** Assume that a set-valued stochastic process \((F_t, F_t)\) has closed convex values and for every \( z \in \mathbb{R}^n \), \( s \geq 0 \), and \( A \in F_s \), the real-valued function \( t \mapsto E(\chi_A d(z, F_t)) \) is right-hand usc at \( s \). Then \((F_t, F_t)\) has a \( L^1 \)-right-hand continuous selection \((f_t, f_t)\).

**Proof:** Define a set-valued mapping \( G: [0, + \infty) \to L^1(\Omega, F_t, \mathbb{R}^n) \) by

\[
G(t) = \{ \varphi \in L^1(\Omega, F_t, \mathbb{R}^n) : \varphi \text{ is } \mathcal{F}_t\text{-measurable selection of } F_t \}.
\]

Based on the assumption \( E(d(z, F_t)) < + \infty \) for each \( t \geq 0 \), the mapping \( G \) has non-empty values by virtue of the Kuratowski and Ryll-Nardzewski measurable selection theorem (see e.g., [5, Theorem 5.1]). Moreover, the sets \( G(t) \) are closed and convex because the set-valued random variables \( F_t \) have closed, convex values. If we equip \([0, + \infty)\) with the arrow topology \( \tau_\rightarrow \) (i.e., the topology generated by the intervals \([s, t), 0 \leq s < t\)), then it follows from the assumptions that \( G: [0, + \infty) \to L^1(\Omega, F_t, \mathbb{R}^n) \) is a lower semicontinuous, set-valued mapping. Indeed, it suffices to show that

\[
d(\varphi, G(t)) = \inf_{\psi \in G(t)} E( | \varphi - \psi | ) \to 0 \quad \text{as } t \uparrow s \quad \text{for any } \varphi \in G(s), \ s \geq 0.
\]

Since \( \varphi \) is \( \mathcal{F}_t \)-measurable for \( t \geq s \), as a consequence of Kuratowski and Ryll-Nardzewski selection theorem, we have that

\[
d(\varphi, G(t)) = E(d(\varphi, F_t))
\]

for \( t \geq s \), (see Hiai and Umegaki [4, Theorem 2.2] and Rybiński [8, Lemma 6]). But by virtue of Lemma 2 we have that \( E(d(\varphi, F_t)) \to 0 \) as \( t \downarrow s \). This shows that \( G \) is lower semicontinuous on \(([0, + \infty), \tau_\rightarrow)\). Since \(([0, + \infty), \tau_\rightarrow)\) is a Lindelöf space, hence paracompact (see Engelking [2]), we can then apply the Michael continuous selection theorem to \( G \) ([7, Theorem 3.2'']) and get a continuous mapping \( g: [0, + \infty) \to L^1(\Omega, F_t, \mathbb{R}^n) \) such that \( g(t) \in G(t) \) for all \( t \geq 0 \). Obviously, continuity with respect to \( \tau_\rightarrow \) means the right-hand continuity of \( g \). We can then define the stochastic process \((f_t)_{t \geq 0}\) by \( f_t(\omega) = g(t)(\omega) \). Clearly, a selection \((f_t)\) is \( \mathcal{F}_t \)-adapted. Since \( E(| f_t - f_s |) = E(| g(t) - g(s) |) \to 0 \) as \( t \downarrow s \), then by the Chebyshev inequality, \( P(| f_t - f_s | > \epsilon) \to 0 \) as \( t \to s \). Thus, \((f_t, \mathcal{F}_t)\) is stochastically right-hand continuous.

For the proof of the next selection theorem, we will need also the following consequence of Levy's martingale convergence theorem.

**Proposition 1:** \( \mathcal{F}_t = \mathcal{F}_{t-} \) if and only if the function \( s \mapsto E(\varphi | \mathcal{F}_s) \) is \( P \)-almost everywhere left-hand continuous at \( t \) for each \( \mathcal{F}_t \)-measurable \( \varphi \) such that \( E(| \varphi |) < + \infty \). Analogously, \( \mathcal{F}_t = \mathcal{F}_{t+} \) if and only if the function \( s \mapsto E(\varphi | \mathcal{F}_s) \) is \( P \)-almost everywhere right-hand continuous at \( t \) for each \( \mathcal{F} \)-measurable \( \varphi \) such that \( E(| \varphi |) < + \infty \).

**Proof:** If \( \mathcal{F}_t = \mathcal{F}_{t-} \), then by Levy's theorem (see Liptser and Shirayev [6, p. 24])

...
we have that $E(\varphi | \mathcal{F}_{s_n}) \rightarrow E(\varphi | \mathcal{F}_t)$ whenever $s_n \uparrow t$. Conversely, observe that for $A \in \mathcal{F}_t$, $E(\chi_A | \mathcal{F}_{s_n}) \rightarrow E(\chi_A | \mathcal{F}_t)$ by Levy's theorem whenever $s_n \uparrow t$. On the other hand, by assumption $E(\chi_A | \mathcal{F}_{s_n}) \rightarrow E(\chi_A | \mathcal{F}_t) = \chi_A$, thus $\chi_A = E(\chi_A | \mathcal{F}_{t-})$ $P$-almost everywhere. Therefore, for $B = (E(\chi_A | \mathcal{F}_{t-}))^{-1}(1) \in \mathcal{F}_{t-}$, $P((A \setminus B) \cup (B \setminus A)) = 0$. Since all $P$-null sets are in $\mathcal{F}_{t-}$, we conclude that $A \in \mathcal{F}_{t-}$. The analogous statement regarding $\mathcal{F}_{t+}$ can be verified in the same way.

**Theorem 3:** Let $\mathcal{F}_t = \mathcal{F}_{t-}$ for each $t > 0$. Assume that a set-valued stochastic process $(F_t, \mathcal{F}_t)$ has closed values and for every $z \in \mathbb{R}^n$, $s \geq 0$, $A \in \mathcal{F}_s$, the real-valued function $t \mapsto E(\chi_A d(z, F_t))$ is use at $s$. Assume also that $P$ is nonatomic or $(F_t, \mathcal{F}_t)$ has convex values. Then $(F_t, \mathcal{F}_t)$ has an $L^1$-continuous selection $(f_t, \mathcal{F}_t)$.

**Proof:** We consider $[0, +\infty)$ with the usual topology and will show that $G$ (defined in the proof of Theorem 2) is lower semicontinuous. The right-hand lower semicontinuity can be proved exactly in the same way as in Theorem 2, so it suffices to show that for fixed $s > 0$, $\varphi \in G(s)$, we have $d(\varphi, G(t)) \rightarrow 0$ as $t \uparrow s$. But for $t < s$, we have

$$d(\varphi, G(t)) \leq E(\varphi - E(\varphi | \mathcal{F}_t) |) + d(E(\varphi | \mathcal{F}_t), G(t))$$

$$= E(\varphi - E(\varphi | \mathcal{F}_t) |) + d(E(\varphi | \mathcal{F}_t), F_t))$$

$$\leq E(\varphi - E(\varphi | \mathcal{F}_t) |) + E(\varphi - \varphi |) + d(\varphi, F_t)).$$

By Proposition 1 we have $E(\varphi - E(\varphi | \mathcal{F}_t) |) \rightarrow 0$ as $t \uparrow s$, and by Lemma 2 we have $E(d(\varphi, F_t)^2) \rightarrow 0$ as $t \uparrow s$. Therefore $G$ is a lower semicontinuous set-valued mapping with closed values. Suppose now that $P$ is nonatomic. Clearly, the sets $G(t)$ are decomposable (i.e., $\varphi \chi_A + \psi \chi_{\Omega \setminus A} \in G(t)$ whenever $\varphi, \psi \in G(t)$ and $A \in \mathcal{F}_t$). We can apply the Antosiewicz-Cellina continuous selection theorem (see Bressan and Colombo [1, Theorem 3]) to $G$, and get a continuous mapping $g: [0, +\infty) \rightarrow L^1(\Omega, \mathcal{F}, R^n)$ such that $g(t) \in G(t)$ for all $t \geq 0$. If $(F_t, \mathcal{F}_t)$ has convex values, as in the proof of Theorem 2, we get a continuous selection $g$ applying Michael's theorem. Thus, the stochastic process $(f_t)$ defined by $f_t(\omega) = g(t)(\omega)$ has desired properties.

If we assure the continuity of the conditional expectation operator $t \mapsto E(\varphi | \mathcal{F}_t)$, then we can extend Hess' result [3, Theorem 3.2] on the martingale selection of discrete time set-valued martingale and obtain a continuous martingale selection result. A set-valued process $(F_t, \mathcal{F}_t)$ is a set-valued martingale if

$$\{ \varphi \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : \varphi \text{ is } \mathcal{F}_s\text{-measurable selection of } F_s \} \text{ is a set-valued martingale if}$$

$$= \text{cl}\{E(\varphi | \mathcal{F}_s) : \varphi \text{ is } \mathcal{F}_t\text{-measurable selection of } \mathcal{F}_t \}$$

for any $0 \leq s \leq t$, (see Hiai and Umegaki [4], Hess [3]). We propose the following continuous time version of Hess' theorem.

**Proposition 2:** Let $(F_t, \mathcal{F}_t)$ be a set-valued martingale. If for every $t \geq 0$ we have $\mathcal{F}_t = \mathcal{F}_{t-}$, then $(F_t, \mathcal{F}_t)$ admits a martingale selection $(f_t, \mathcal{F}_t)$ with $P$-almost all paths left-hand continuous. If for every $t \geq 0$ we have $\mathcal{F}_t = \mathcal{F}_{t+}$, then $(F_t, \mathcal{F}_t)$ admits a martingale selection $(f_t, \mathcal{F}_t)$ with $P$-almost all paths right-hand continuous.

**Proof:** Consider the discrete time set-valued martingale $(F_n)_n = 0, \ldots$ obtained
from \((F_t, \mathcal{F}_t)\) by taking \(t = 0,1,\ldots\). By the Hess result, \((F_n)\) has a martingale selection \((f_n)\) (i.e., there exists a sequence of \(\mathcal{F}_n\)-measurable mappings \(f_n: \Omega \to \mathbb{R}^n\) such that \(f_n\) is a selection of \(F_n\) and \(f_n = E(f_{n+1} | \mathcal{F}_n)\) for \(n = 0,1,\ldots\)). For \(t \in [0, +\infty) \setminus \{0,1,2,\ldots\}\) we define \(f_t: \Omega \to \mathbb{R}^n\) by \(f_t = E(f_n | \mathcal{F}_t)\) where \(n - 1 < t < n\). Clearly, \((f_t)\) is a martingale selection of \(F\). By Proposition 1, \((f_t)\) has \(P\)-almost all paths left-hand (respectively, right-hand) continuous.

References