AN ITERATIVE ALGORITHM ON FIXED POINTS OF RELAXED LIPSCHITZ OPERATORS

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Fixed points of Lipschitzian relaxed Lipschitz operators based on a generalized iterative algorithm are approximated.

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1. Introduction

Recently, Wittman [6, Theorem 2], using an iterative procedure

\[ x_n = (1 - a_n)x_0 + a_nTx_{n-1} \quad \text{for } n \geq 1, \]

approximated fixed points of nonexpansive mappings \( T: K \to K \) from a nonempty closed convex subset \( K \) of a real Hilbert space \( H \) into itself, where \( x_0 \) is an element of \( K \) and \( \{a_n\} \) is an increasing sequence in \([0, 1)\) such that

\[ \lim_{n \to \infty} a_n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - a_n) = \infty. \]

This result refines a number of results including [1].

Here our aim is to approximate the fixed points of Lipschitzian relaxed Lipschitz operators in a Hilbert space setting. As such, the iterative algorithm (1) is not suitable for our purpose, so we apply a modified iterative algorithm which reduces to (1).

Let \( H \) be a Hilbert space and \( \langle u, v \rangle \) and \( \| u \| \) denote, respectively, the inner product and norm on \( H \) for \( u, v \) in \( H \).

An operator \( T: H \to H \) is said to be relaxed Lipschitz if, for all \( u, v \) in \( H \), there exists a constant \( r > 0 \) such that

\[ \langle Tu - Tv, u - v \rangle \leq -r \| u - v \|^2. \]
The operator $T$ is called Lipschitz continuous (or Lipschitzian) if there exists a constant $s > 0$ such that

$$\| Tu - Tv \| \leq s \| u - v \| \text{ for all } u, v \text{ in } H. \quad (4)$$

Next, we consider the main result on the approximation of the fixed points of Lipschitzian relaxed Lipschitz operators using a modified iterative algorithm which contains a number of iterative schemes including those considered by the author [4, 5] as special cases.

2. The Main Result

**Theorem 1:** Let $H$ be a real Hilbert space and $K$ be a nonempty closed convex subset of $H$. Let $T : K \to K$ be a relaxed Lipschitz and Lipschitz continuous operator on $K$. Let $r \geq 0$ and $s \geq 1$ be constants for relaxed Lipschitzity and Lipschitz continuity of $T$, respectively. Let $F = \{ x \in K : Tx = x \}$ be nonempty, and let $\{ a_n \}$ be a sequence in $[0, 1]$ such that

$$\sum_{n=0}^{\infty} a_n = \infty \text{ for all } n \geq 0. \quad (5)$$

Then for any $x_0$ in $K$ the sequence $\{ x_n \}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n[(1 - t)x_n + tTx_n] \text{ for } n \geq 0, \quad (6)$$

for $0 < k = ((1 - t)^2 - 2t(1 - t)^2 + t^2s^2)^{1/2} < 1$ for all $t$ such that $0 < t < 2(1 + r)/(1 + 2r + s^2)$ and $r \leq s$, converges to an element of $F$.

For $\{ a_n \} = 1$, Theorem 1 reduces to:

**Corollary 1:** Let $T : K \to K$ be relaxed Lipschitz and Lipschitz continuous. Let $F = \{ x \in K : Tx = x \}$ be a nonempty set. Then, for $x_0$ in $K$, the sequence $\{ x_n \}$ generated by an iterative algorithm

$$x_{n+1} = (1 - t)x_n + tTx_n \quad (7)$$

for $0 < t < 2(1 + r)/(1 + 2r + s^2)$ converges to a unique fixed point of $T$.

**Proof of Theorem 1:** For an element $z$ in $F$, we have

$$\| x_{n+1} - z \| = \| (1 - a_n)x_n + a_n[(1 - t)x_n + tTx_n] - z \|$$

$$\leq (1 - a_n) \| (x_n - z) \| + a_n \| (1 - t)(x_n - z) + t(Tx_n - Tz) \|.$$

Using the relaxed Lipschitzity and Lipschitz continuity of $T$, we find that

$$\| t(Tx_n - Tz) + (1 - t)(x_n - z) \|^2$$

$$= (1 - t)^2 \| x_n - z \|^2 + 2t(1 - t)(Tx_n - z, x_n - z) + t^2 \| Tx_n - z \|^2$$

$$\leq (1 - t)^2 \| x_n - z \|^2 - 2t(1 - t)r \| x_n - z \|^2 + t^2s^2 \| x_n - z \|^2.$$
It follows that
\[
\|x_{n+1} - z\| \leq (1 - a_n + a_n((1 - t)^2 - 2t(1-t)r + t^2s^2)^{1/2}) \|x_n - z\|
\]
\[
= (1 - (1-k)a_n) \|x_n - z\|
\]
\[
\leq \prod_{j=0}^{n} (1 - (1-k)a_j) \|x_0 - z\|,
\]
where \(0 < k = ((1-t)^2 - 2t(1-t)r + t^2s^2)^{1/2} < 1\) for all \(t\) such that \(0 < t < 2(1+r)/(1+2r+s^2)\) and \(r \leq s\).

Since \(\sum_{j=0}^{\infty} a_j\) diverges and \(k < 1\), \(\lim_{n \to \infty} \sum_{j=0}^{n} (1-(1-k)a_j) = 0\) and, as a result, \(\{x_n\}\) converges strongly to \(z\). This completes the proof.

References