Majorization for A Subclass of $\beta$-Spiral Functions of Order $\alpha$ Involving a Generalized Linear Operator

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1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}$$

which are analytic in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Let $f$ and $g$ be analytic in $U$. Then, we say that function $f$ is subordinate to $g$ if there exists a Schwarz function $\omega(z)$, analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z)), z \in U$ (see [1]). We denote this subordination by

$$f \prec g \quad (z \in U).$$
Further, \( f \) is said to be quasi subordinate to \( g \) if there exists an analytic function \( \varphi(z) \) such that \( f(z)/\varphi(z) \) is analytic in \( U \),

\[
\frac{f(z)}{\varphi(z)} \lesssim g(z), \quad (z \in U) \tag{1.3}
\]

and \( |\varphi(z)| \leq 1 \). Note that the quasi subordination (1.3) is equivalent to

\[
f(z) = \varphi(z)g(\omega(z)) \tag{1.4}
\]

where \( |\varphi(z)| \leq 1 \) and \( |\omega(z)| \leq |z| < 1 \) (see [2]). If \( \varphi(z) = 1 \), then (1.3) becomes (1.2).

Let functions \( f \) and \( g \) be analytic functions in \( U \). If \( |f(z)| \leq |g(z)| \), then there exists a function \( \varphi \) analytic in \( U \) such that \( |\varphi(z)| \leq 1 \) in \( U \), for which

\[
f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.5}
\]

In this case, we say that \( f \) is majorized by \( g \) in \( U \) (see [3]), and we write

\[
f(z) \lesssim g(z) \quad (z \in U). \tag{1.6}
\]

If we take \( \omega(z) = z \) in (1.4), then the quasi subordination (1.3) becomes the majorization (1.6).

Also, let \( S \) denote the subclass of \( A \) consisting of all functions which are univalent in \( U \).

In [4], Robertson introduced star-like functions of order \( \alpha \) on \( U \).

\textit{Definition 1.1.} Let \( 0 \leq \alpha < 1 \) and \( f \in A \); then, \( f \) is a star-like function of order \( \alpha \) on \( U \) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U). \tag{1.7}
\]

Let \( S^*(\alpha) \) denote the whole star-like functions of order \( \alpha \) in \( U \).

Spaček [5] extended the class of \( S^* \) and obtained the class of \( \beta \)-spiral-like functions. In the same article, the author gave an analytical characterization of spirallikeness of type \( \beta \) on \( U \).

\textit{Definition 1.2.} Let \( -\pi/2 < \beta < \pi/2 \) and \( f \in A \); then, \( f \) is \( \beta \)-spiral-like function on \( U \) if and only if

\[
\Re \left\{ e^{i\beta} \frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in U). \tag{1.8}
\]

We denote the whole \( \beta \)-spiral-like functions in \( U \) by \( S^*_\beta \).
Finally, Libera [6] introduced and studied the class of $\beta$-spiral-like functions of order $\alpha$.

**Definition 1.3.** Let $0 \leq \alpha < 1$, $-\pi/2 < \beta < \pi/2$ and $f \in A$; then, $f$ is $\beta$-spiral function of order $\alpha$ if and only if

$$\Re \left\{ e^{i\beta} \frac{z f'(z)}{f(z)} \right\} > \alpha \cos \beta \quad (z \in U).$$

(1.9)

We denote the whole $\beta$-spiral-like functions of order $\alpha$ in $U$ by $S^*_\beta(\alpha)$.

In particular, we consider the convolution with function $\phi(a, c)$ defined by

$$L(a, b) f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1},$$

(1.10)

where $a \in \mathbb{C}$, $b \neq 0, -1, -2, \ldots$, and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a + 1) \cdots (a + n - 1), & n = \{1, 2, 3, \ldots\}. \end{cases}$$

(1.11)

Function $\phi(a, c)$ is an incomplete beta-function related to the Gauss hypergeometric function by

$$\phi(a, c; z) = z_2 F_1(1, a; c; z).$$

(1.12)

It has an analytic continuation to the $z$-plane cut along the positive real line from $1$ to $\infty$. We note that $\phi(1; z) = z/(1 - z)^a$ and $\phi(2, 1; z)$ are the Koebe functions.

Carlson and Shaffer [7] defined a convolution operator on $A$ involving an incomplete beta-function as

$$L(a, b) f(z) = \phi(a, c; z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_n^{(n+1)} z^{n+1}.$$  

(1.13)

**Definition 1.4.** Let function $F$ be given by

$$F(m, \ell, \lambda) = \sum_{n=0}^{\infty} \left( \frac{1 + \ell + \lambda n}{1 + \ell} \right)^m z^{n+1},$$

(1.14)

where $\ell, \lambda \geq 0$ and $m \in \mathbb{Z}$. The generalized linear operator $L(m, \ell, \lambda, a, c) : A \to A$ is given as

$$L(m, \ell, \lambda, a, b) f(z) = z + \sum_{n=1}^{\infty} \left( \frac{1 + \ell + \lambda n}{1 + \ell} \right)^m \frac{(a)_n}{(c)_n} a_n^{(n+1)} z^{n+1}.$$  

(1.15)
We note here some special cases.

1. $L(0, \ell, \lambda, a, b)f(z) = L(a, b)f(z)$ is the Carlson and Shaffer operator [7].
2. $L(0, \ell, \lambda, \beta + 1, 1)f(z)$, $\beta \in \mathbb{N}_0$, is the Ruscheweyh derivative [8].
3. $L(m, 0, \lambda, 1, 1)f(z)$, $m \in \mathbb{N}_0$, is the Al-Oboudi operator [9].
4. $L(m, 0, \lambda, a, b)f(z)$ is the linear operator introduced by Al-Refai and Darus [10].
5. $F(m, \ell, \lambda)$, $m \in \mathbb{N}_0$, is the generalized multiplier transformation which was introduced and studied by Catbas [11].
6. $F(m, \ell, 1), m \in \mathbb{N}_0$, is the multiplier transformation which was introduced and studied by Cho and Srivastava [12] and Cho and Kim [13].

**Remark 1.5.** It follows from the above definition that

$$z(L(m, \ell, \lambda, a, c)f(z))' = aL(m, \ell, \lambda, a + 1, c)f(z) - (a - 1)L(m, \ell, \lambda, a, c)f(z) \quad (z \in \mathcal{U}).$$

(1.16)

We introduce the class $S^\ast_p(m, \ell, \lambda, a, c, \alpha)$ as follows.

**Definition 1.6.** Let $a \in \mathbb{C}$, $c \neq 0, -1, -2, ..., \ell$, $\lambda \geq 0$, $m \in \mathbb{Z}$, $0 \leq a < 1$, $-\pi/2 < \beta < \pi/2$, and $f \in A$; then, one has $S^\ast_p(m, \ell, \lambda, a, c, \ell, \lambda, \alpha)$ if and only if

$$\mathfrak{R}\left\{e^{i\phi}z(L(a, c)f(z))' \right\} > a \cos \beta.$$

(1.17)

Obviously, when $a = c = 1$ and $m = 0$ we obtain $f \in S^\ast_p(\alpha)$, when $a = c = 1$ and $m = \beta = 0$, we obtain that $f(z)$ is a star-like function of order $\alpha$ on $\mathcal{U}$, and also when $a = c = 1$ and $m = \alpha = 0$, we obtain that $f(z)$ is spiral-like function of type $\beta$ on $\mathcal{U}$.

Biernacki [14] in 1936 obtained the first results of majorization-subordination theory. He showed that, if $g(z) \in S$ and $f(z) < g(z)$ in $\mathcal{U}$, then $f(z) \ll g(z)$ in $|z| \leq (1/4)$. Goluzin [15] improved the result and Shah [16] obtained the complete solution for $S$ by showing that $f(z) \ll g(z)$ in $|z| \leq (3 - \sqrt{5})/2$ and that the result is the best possible. A majorization problem for star-like functions has been given by MacGregor [3]. Also, majorization problem for star-like functions of complex order has recently been investigated by Altintas et al. [17].

The main object of this paper is to investigate the problem of majorization of the class $S^\ast_p(\ell, \lambda, a, c, \alpha)$ defined by a generalized linear operator.

In order to prove our main theorem we need the following lemma.

**Lemma 1.7** (see [18]). Let $\varphi(z)$ be analytic in $\mathcal{U}$ satisfying $|\varphi(z)| \leq 1$ for $z \in \mathcal{U}$. Then,

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$ 

(1.18)
2. Main Results

Theorem 2.1. Let function \( f \in A \) and suppose that \( g \in S^*_r(m, \ell, \lambda, a, c, \alpha) \). If \( L(m, \ell, \lambda, a, c) f \) is majorized by \( L(m, \ell, \lambda, a, c) g \) in \( U \), then

\[
|L(m, \ell, \lambda, a + 1) f(z)| \leq |L(m, \ell, \lambda, a + 1) g(z)| \quad (|z| \leq r_1),
\]

where

\[
r_1 = \frac{2 + |a| + |2(1 - \alpha) \cos \beta - ae^{i\beta}|}{2|2(1 - \alpha) \cos \beta - ae^{i\beta}|} - \frac{\sqrt{\Theta(a, \alpha, \beta)}}{2|2(1 - \alpha) \cos \beta - ae^{i\beta}|}.
\]

\[
\Theta(a, \alpha, \beta) = 4 + |a|^2 + |2(1 - \alpha) \cos \beta - ae^{i\beta}|^2 + 4|a| + 4|2(1 - \alpha) \cos \beta - ae^{i\beta}| - 2|a| |2(1 - \alpha \cos \beta - ae^{i\beta}|.
\]

for \( a \in \mathbb{C} \), \( c \neq 0, -1, -2, \ldots, \ell, \lambda \geq 0, m \in \mathbb{Z}, 0 \leq \alpha < 1, -\pi/2 < \beta < \pi/2, \) and \(|a| \geq |2(1 - \alpha) \cos \beta - ae^{i\beta}|\).

Proof. Since \( g \in S^*_r(m, \ell, \lambda, a, c, \alpha) \), we have

\[
e^{i\beta} \frac{z(L(m, \ell, \lambda, a, c) g(z))'}{L(m, \ell, \lambda, a, c) g(z)} = \frac{1 + (1 - 2\alpha) \omega}{1 - \omega} \cos \beta + i \sin \beta,
\]

where \( \omega \) is analytic in \( U \), with \( \omega(0) = 0 \) and

\[
|\omega| \leq |z| < 1 \quad (z \in U).
\]

By using (1.16) in (2.4), we get

\[
e^{i\beta} \frac{[a L(m, \ell, \lambda, a + 1) g(z) - (a - 1)L(m, \ell, \lambda, a, c) g(z)]}{L(m, \ell, \lambda, a, c) g(z)} = \frac{1 + (1 - 2\alpha) \omega}{1 - \omega} \cos \beta + i \sin \beta.
\]

Hence,

\[
\frac{L(m, \ell, \lambda, a + 1) g(z)}{L(m, \ell, \lambda, a, c) g(z)} = \frac{ae^{i\beta} + (2(1 - \alpha) \cos \beta - ae^{i\beta}) \omega}{ae^{i\beta}(1 - \omega)},
\]

which, in view of (2.5), immediately yields the inequality

\[
|L(m, \ell, \lambda, a, c) g(z)| \leq \frac{|e^{i\beta}| |a| (1 + |z|)}{|a| - |2(1 - \alpha \cos \beta - ae^{i\beta})| |z|} |L(m, \ell, \lambda, a + 1) g(z)|.
\]
Next, since $L(m, \ell, \lambda, a, c)f$ is majorized by $L(m, \ell, \lambda, a, c)g$ in $U$, from (1.5) we have

$$z(L(m, \ell, \lambda, a, c)f(z))' = z\phi'(z)L(m, \ell, \lambda, a, c)g(z) + z\phi(z)(L(m, \ell, \lambda, a, c)g(z))'.$$

(2.9)

Also, by using (1.6) in (2.11), we get

$$aL(m, \ell, \lambda, a + 1, c)f(z) - (a - 1)L(m, \ell, \lambda, a, c)f(z)
= z\phi'(z)L(m, \ell, \lambda, a, c)g(z) + \phi(z)[aL(m, \ell, \lambda, a + 1, c)g(z) - (a - 1)L(m, \ell, \lambda, a, c)g(z)];$$

(2.10)

then, we have

$$L(m, \ell, \lambda, a + 1, c)f(z) = \frac{1}{a}z\phi'(z)L(m, \ell, \lambda, a, c)g(z) + \phi(z)L(m, \ell, \lambda, a + 1, c)g(z).$$

(2.11)

Thus, by Lemma 1.7, since the Schwarz function $\phi$ satisfies the inequality in (1.8) and using (2.8) in (2.11), we get

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{(1 - |\phi(z)|^2)|z|}{(1 - |z|)(|a| - |2(1 - \alpha)\cos \beta - a\phi^\beta||z|)} \times |L(m, \ell, \lambda, a + 1, c)g(z)| + |\phi(z)| \cdot |L(m, \ell, \lambda, a + 1, c)g(z)|.$$  

(2.12)

Hence,

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{(1 - |\phi(z)|^2)|z| + (1 - |z|)(|a| - |2(1 - \alpha)\cos \beta - a\phi^\beta||z|)|\phi(z)|}{(1 - |z|)(|a| - |2(1 - \alpha)\cos \beta - a\phi^\beta||z|)} \times |L(m, \ell, \lambda, a + 1, c)g(z)|, $$

(2.13)

which, upon setting

$$|z| = r, \quad |\phi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

(2.14)

yields

$$|L(m, \ell, \lambda, a + 1, c)f(z)| \leq \frac{\theta(\rho)}{(1 - r)(|a| - |2(1 - \alpha)\cos \beta - a\phi^\beta|r)} |L(m, \ell, \lambda, a + 1, c)g(z)|.$$ 

(2.15)
where function $\theta(\rho)$ defined by

$$\theta(\rho) = (1 - \rho^2)r + (1 - r)(|a| - |2(1 - \alpha)\cos \beta - ae^{i\theta}|r)\rho$$

(2.16)

takes on its maximum value at $\rho = 1$ with

$$r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta) = \max\{r \in [0, 1] : \psi(r, \rho) \leq 1, \ \forall \rho \in [0, 1]\}$$

(2.17)

where

$$\psi(r, \rho) = \frac{\theta(\rho)}{(1 - r)(|a| - |2(1 - \alpha)\cos \beta - ae^{i\theta}|r)}$$

(2.18)

then, we have

$$\frac{\theta(\rho)}{(1 - r)(|a| - |2(1 - \alpha)\cos \beta - ae^{i\theta}|r)} \leq 1.$$  

(2.19)

A simple calculus in (2.19) is equivalent to

$$-(1 + \rho)r + (1 - r)(|a| - |2(1 - \alpha)\cos \beta - ae^{i\theta}|r) \geq 0,$$

(2.20)

while the inequality in (2.19) takes its minimum value at $\rho = 1$, that is,

$$|2(1 - \alpha)\cos \beta - ae^{i\theta}|r^2 - (2|a| + |2(1 - \alpha)\cos \beta - ae^{i\theta}|r + |a| \geq 0,$$

(2.21)

for all $r \in [0, r_1]$, where $r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta)$ given in (2.2) holds true for $|z| \leq r(m, \ell, \lambda, a, c, \alpha, \beta)$, which proves the conclusion (2.1).

Putting $m = a = \beta = 0$ in Theorem 2.1, we obtain the following result.

**Corollary 2.2.** Let function $f \in A$ and suppose that $g \in S^a(a, c)$. If $L(a, c)g$ in $U$, then

$$|L(a + 1, c)f(z)| \leq |L(a + 1, c)g(z)| (|z| \leq r_2 = r(a, c)),$$

(2.22)

where

$$r(a, c) = \frac{3 + |a| + |2 - a|}{2|2 - a|} - \sqrt{\frac{4 + |2 - a|^2 - 2|a||2 - a| + 4|a| + |a|^2}{2|2 - a|}}.$$  

(2.23)

Further, putting $a = c = 1$ and $m = 0$ in Theorem 2.1, we obtain the result of Altintas et al. [17].
Corollary 2.3. Let function \( f \in A \) and suppose that \( g \in S^{*}(\alpha) = S^{*}_{\rho}(\alpha) \), where \( 0 \leq \alpha < 1 \) and \( -\pi/2 < \beta < \pi/2 \). If \( f \) is majorized by \( g \) in \( U \), then

\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq r_3 = r(\alpha, \beta)),
\]

where

\[
r(\alpha, \beta) = \frac{3 + |2(\alpha - 1)e^{i\beta} - 1| + \sqrt{9 + |2(\alpha - 1)e^{i\beta} - 1|^{2} + 2|2(\alpha - 1) - 1|^{2}}}{2|2(\alpha - 1)e^{i\beta} - 1|}.
\]

Putting \( \beta = 0 \) in Corollary 2.3, we obtain the result as follows.

Corollary 2.4. Let function \( f \in A \) and suppose that \( g \in S^{*}(\alpha) \), where \( 0 \leq \alpha < 1 \). If \( f \) is majorized by \( g \) in \( U \), then

\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq r_4 = r(\alpha)),
\]

where

\[
r(\alpha) = \frac{3 + |1 - 2\alpha| + \sqrt{9 + |1 - 2\alpha|^{2} + 2|2(\alpha - 1) - 1|^{2}}}{2|1 - 2\alpha|}.
\]

Also, putting \( \alpha = \beta = 0 \) in Corollary 2.3, we obtain the result of MacGregor [3].

Corollary 2.5. Let function \( f \in A \) and suppose that \( g \in S^{*}(0) \). If \( f \) is majorized by \( g \) in \( U \), then

\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq 2 - \sqrt{3}).
\]

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References


