Research Article

Viscosity Approximation Methods and Strong Convergence Theorems for the Fixed Point of Pseudocontractive and Monotone Mappings in Banach Spaces

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Suppose that $C$ is a nonempty closed convex subset of a real reflexive Banach space $E$ which has a uniformly Gateaux differentiable norm. A viscosity iterative process is constructed in this paper. A strong convergence theorem is proved for a common element of the set of fixed points of a finite family of pseudocontractive mappings and the set of solutions of a finite family of monotone mappings. And the common element is the unique solution of certain variational inequality. The results presented in this paper extend most of the results that have been proposed for this class of nonlinear mappings.

1. Introduction

Let $E$ be a real Banach space with dual $E^*$. A normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$Jx = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between $E$ and $E^*$. It is well known that $E$ is smooth if and only if $J$ is single valued, and $J$ is uniformly continuous on bounded subsets of $E$. Moreover, if $E$ is reflexive and strictly convex Banach space with a strictly convex dual then $J^{-1}$ is single valued, one-to-one surjective, and it is the duality mapping from $E^*$ into $E$, and then $JJ^{-1} = I_{E^*}$ and $JJ^{-1} = I_E$.

A mapping $A : D(A) \subset E \rightarrow E^*$ is said to be monotone if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0.$$  \hspace{1cm} (2)

A mapping $A : D(A) \subset E \rightarrow E^*$ is said to be strongly monotone, if there exists a positive real number $\alpha > 0$ such that for all $x, y \in D(A)$

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2.$$  \hspace{1cm} (3)

Obviously, the class of monotone mappings includes the class of the $\alpha$-inverse strongly monotone mappings. Let $C$ be closed convex subset of Banach space $E$. A mapping $T$ is said to be pseudocontractive if, for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2.$$  \hspace{1cm} (4)

A mapping $T$ is said to be $\kappa$-strictly pseudo-contractive, if, for any $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a constant $0 \leq \kappa \leq 1$ such that

$$\langle x - y - (Tx - Ty), j(x - y) \rangle \geq \kappa \|I - T\| x - (I - T) y\|^2.$$  \hspace{1cm} (5)

A mapping $f : C \rightarrow C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C.$$  \hspace{1cm} (7)
Let $E$ be a real Banach space with dual $E^*$. The norm on $E$
 is said to be uniformly Gateaux differentiable if for each $y \in
 S_1(0) := \{x \in E : \|x\| = 1\}$ the limit $\lim_{t \to 0} ([x + t y] - \|x\|)/t$
 exists uniformly for $x \in S_1(0)$.

For finding an element of the set of fixed points of the non-expansive
mappings, Halpern [1] was the first to study the convergence
of the scheme in 1967:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n).$$  \hspace{1cm} (8)

Viscosity approximation methods are very important
because they are applied to convex optimization, linear pro-
gramming, monotone inclusions, and elliptic differential
equations. In Hilbert spaces, many authors have studied the
fixed points problems of the fixed points for the non-expans-
ive mappings and monotone mappings by the viscosity
approximation methods, and obtained a series of good results
(see [2–17]).

In 2000, Moudifi [18] introduced the viscosity approxi-
mation methods and proved the strong convergence of the
following iterative algorithm in Hilbert spaces under some
suitable conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n).$$  \hspace{1cm} (9)

Suppose that $A$ is monotone mapping from $C$ into $E$.
The classical variational inequality problem is formulated as
finding a point $u \in C$ such that $(v - u, Au) \geq 0$, for all
$v \in C$. The set of solutions of variational inequality problems
is denoted by $VI(C, A)$.

Takahashi and Toyoda [19, 20] introduced the follow-
ing scheme in Hilbert spaces and studied the weak and strong
convergence theorem of the elements of $F(T) \cap VI(C, A)$,
respectively, under different conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_P(x_n - \lambda_n x_n),$$  \hspace{1cm} (10)

where $T$ is non-expansive mapping and $A$ is $\alpha$-inverse strong
monotone operator.

Recently, Zegeye and Shahzad [21] introduced the follow-
ing algorithm and obtained the strong convergence theorem
but still in Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{n, r_n} x_n,$$  \hspace{1cm} (11)

where $T_{n, r_n}$ are nonexpansive mappings.

Our concern now is the following: is it possible to con-
struct a new sequence in Banach spaces which converges
strongly to a common element of fixed points of a finite family
of pseudocontractive mappings and the solution set of a vari-
ational inequality problems for finite family of monotone
mappings?

2. Preliminaries

In the sequel, we will use the following lemmas.

Lemma 1 (see, e.g., [5]). Let \{\alpha_n\} be a sequence of nonnegative
real numbers satisfying the following relation:

$$\alpha_{n+1} \leq (1 - \theta_n) \alpha_n + \sigma_n, \quad n \geq 0,$$  \hspace{1cm} (12)

where \{\theta_n\} is a sequence in $(0, 1)$ and \{\sigma_n\} is a real sequence such
that

(i) $\sum_{n=0}^\infty \theta_n = \infty$;

(ii) $\limsup_{n \to \infty} \sigma_n/\theta_n \leq 0$ or $\sum_{n=0}^\infty \sigma_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2 (see, e.g., [10]). Let $C$ be a nonempty, closed, and
convex subset of uniformly smooth strictly convex real Banach
space $E$ with dual $E^*$. Let $A : C \to E^*$ be a continuous
monotone mapping; define mapping $F_r$ as follows: $x \in E, r \in (0, \infty)$

$$F_r(x) = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \right\}.$$  \hspace{1cm} (13)

Then the following hold:

(i) $F_r$ is well defined and single valued;

(ii) $F_r$ is a firmly non-expansive mapping; that is,

$$\langle F_r x - F_r y, JF_r x - JF_r y \rangle \leq \langle F_r x - F_r y, Jx - Jy \rangle;$$

(iii) $F(F_r) = VI(C, A)$;

(iv) $VI(C, A)$ is closed and convex.

Lemma 3 (see, e.g., [22]). Let $C$ be a nonempty closed convex
subset of uniformly smooth strictly convex real Banach space
$E$. Let $T : C \to E$ be a continuous strictly pseudococontractive
mapping; define mapping $T_r$, as follows: $x \in E$, $r \in (0, \infty)$

$$T_r(x) = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r) Jz - Jx \rangle \leq 0, \right\}.$$  \hspace{1cm} (14)

Then the following hold:

(i) $T_r$ is single valued;

(ii) $T_r$ is a firmly nonexpansive mapping; that is,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(iii) $F(T_r) = F(T)$;

(iv) $F(T)$ is closed and convex.

Lemma 4 (see, e.g., [23]). Let \{\xi_n\} and \{\zeta_n\} be bounded
sequence in a Banach space, and let \{\beta_n\} be a sequence in $[0, 1]$
which satisfies the following condition:

$$0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n < 1.$$  \hspace{1cm} (15)

Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) \zeta_n, \quad n \geq 0,$$  \hspace{1cm} (16)

$$\lim_{n \to \infty} (\|z_n - \xi_n\| - \|x_n - \xi_n\|) \leq 0.$$  \hspace{1cm} (17)

Then $\lim_{n \to \infty} \|z_n - x_n\| = 0$. 

Lemma 5 (see, e.g., [16]). Let $C$ be a nonempty closed and convex subset of a real smooth Banach space $E$. A mapping $\Pi_C : E \to C$ is a generalized projection. Let $x \in E$; then $x_0 = \Pi_C x$ if and only if
\[
\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C.
\] (17)

Lemma 6. Let $E$ be a real Banach space with dual $E^*$. $J : E \to 2^{E^*}$ is the generalized duality pairing; then, for all $x, y \in E$,
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).
\] (18)

3. Main Results

Let $C$ be a nonempty closed convex and bounded subset of a smooth, strictly convex and reflexive real Banach space $E$ with dual $E^*$. Let $A_i : C \to E^*$, $i = 1, 2, \ldots, m$ be a finite family of continuous monotone mappings, and let $T_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of continuously strictly pseudo-contractive mappings. For the rest of this paper, $T_n$, $\Gamma_n$ and $F_n$ are mappings defined as follows: for $x \in E$, $r_n \in (0, \infty)$,
\[
T_n(x) := \left\{ z \in C : \langle y - z, T_n z \rangle - \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \leq 0, \forall y \in C \right\}.
\] (19)

Consider
\[
F_n(x) := \left\{ z \in C : \langle y - z, A_i z \rangle + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.
\] (20)

Denote $F_1 = \cap_{i=1}^m F(T_i)$, $F_2 = \cap_{i=1}^m F(T_i)$. 

Lemma 7. Let $C$ be a nonempty closed convex and bounded subset of a smooth Banach space $E$, and let $T_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of non-expansive mappings such that $\cap_{i=1}^m F(T_i) \neq \emptyset$. Suppose that $\alpha = \inf \{\alpha_i\} > 0$; then there exists non-expansive mapping $\Gamma : C \to C$ such that $F(\Gamma) = \cap_{i=1}^m F(T_i)$.

Proof. Let $\{\alpha_i\}$ be any sequence of positive real numbers satisfying $\sum_{i=1}^m \alpha_i = 1$ and set $\Gamma = \sum_{i=1}^m \alpha_i T_i$. Since each $T_i$ is non-expansive for any $i \in \{1, 2, \ldots, m\}$, we have that $\Gamma$ is well-defined non-expansive mapping (see, e.g., [20]), and
\[
\|\Gamma x - \Gamma y\| \leq \sum_{i=1}^m \alpha_i \|T_i x - T_i y\| \leq \sum_{i=1}^m \alpha_i \|T_i x - T_i y\| \leq \|x - y\|.
\] (21)

Next, we claim that $F(\Gamma) = \cap_{i=1}^m F(T_i)$.

Clearly $\cap_{i=1}^m F(T_i) \subset F(\Gamma)$. Now, we prove that $F(\Gamma) \subset \cap_{i=1}^m F(T_i)$. Let $x \in F(\Gamma)$ and $p \in \cap_{i=1}^m F(T_i)$. Then
\[
0 = \langle \Gamma x - x, j(x - p) \rangle = \left\langle \sum_{i=1}^m \alpha_i T_i x - x, j(x - p) \right\rangle
\]
\[
= \sum_{i=1}^m \alpha_i \langle T_i x - x, j(x - p) \rangle,
\] (22)

and non-expansivity of each $T_i$ implies that $\langle T_i x - x, j(x - p) \rangle = 0$ for each $i = 1, 2, \ldots, m$, which implies $T_i x = x$ for each $i = 1, 2, \ldots, m$. Therefore, $x \in \cap_{i=1}^m F(T_i)$, and hence $F(\Gamma) = \cap_{i=1}^m F(T_i)$. The proof is complete. \hfill \Box

Theorem 8. Let $C$ be a nonempty closed convex subset of a real reflexive Banach space $E$ which has a uniformly Gateaux differentiable norm. Let $T_i : C \to C$, $i = 1, 2, \ldots, m$, be a finite family of continuous strictly pseudo-contractive mappings, $A_i : C \to E^*$, $i = 1, 2, \ldots, m$, be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0, 1)$. $T_n$ and $F_n$ are defined as (19) and (20), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$
\[
y_n = \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^m \mu_i F_n x_n,
\] (23)
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^m \sigma_i T_{\Gamma_n} y_n,
\]
where $\lambda_n \in [0, 1]$ and $\{\alpha_n\}, \{\beta_n\},$ and $\{\gamma_n\}$ are sequences of non-negative real numbers in $[0, 1]$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1, n \geq 0, \sum_{i=1}^m \mu_i = 1, \sum_{i=1}^m \sigma_i = 1, \mu_i \geq 0, \sigma_i \geq 0$;

(ii) $\lim_{n \to \infty} \alpha_n = 0, \sum_{i=1}^m \sigma_i = \infty$;

(iii) $0 < \lim \inf_{n \to \infty} \beta_n < \lim \sup_{n \to \infty} \beta_n < 1$;

(iv) $\lim \sup_{n \to \infty} r_n > 0, \sum_{i=1}^m |\sigma_i - \mu_i| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $x = \Pi_F f(x)$, and also $x$ is the unique solution of the variational inequality
\[
\langle (f - I)(x), j(y - x) \rangle \leq 0, \quad \forall y \in F.
\] (24)

Proof. First we prove that $\{x_n\}$ is bounded. Take $p \in F$, because $T_{\Gamma_n}$ is non-expansive; then we have that
\[
\|y_n - p\| \leq \lambda_n \|x_n - p\| + (1 - \lambda_n)
\]
\[
\times \sum_{i=1}^m \mu_i \|F_{\Gamma_n} x_n - F_{\Gamma_n} p\| \leq \|x_n - p\|.
\] (25)
For $n \geq 0$, because $T_{fr_{n}}$ and $F_{fr_{n}}$ are nonexpansive and $f$ is contractive, we have from (25) that
\[
\|x_{n+1} - p\| \\
= \|\alpha_n (f(x_n) - p) + \beta_n (\gamma_n - p) + y_n \left( \sum_{i=1}^{m} \sigma_i T_{fr_{n}} y_n - p \right)\| \\
\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\
+ \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\
\leq \rho \alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\
\leq \frac{1 - (1 - \rho) \alpha_n}{\rho} \|x_n - p\| + \alpha_n \|f(p) - p\| \\
\leq \max \left\{ \|x_0 - p\|, \frac{f(p) - p}{1 - \rho} \right\}.
\]
(26)

Therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{fr_{n}} x_n\}, \{T_{fr_{n}} y_n\}$ and $\{y_n\}, \{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$.

Consider
\[
\|y_{n+1} - y_n\| \\
\leq \lambda_{n+1} \|x_{n+1} - x_n\| + (1 - \lambda_{n+1}) \sum_{i=1}^{m} \mu_i \|F_{fr_{n}} x_{n+1} - F_{fr_{n}} x_n\| \\
+ |\lambda_{n+1} - \lambda_n| \|x_n - \sum_{i=1}^{m} \mu_i F_{fr_{n}} x_n\|.
\]
(27)

Let $v_{in} = F_{fr_{n}} x_n, v_{in+1} = F_{fr_{n}} x_{n+1}$; by the definition of mapping $F_{fr_{n}},$ we have that
\[
\langle y - v_{in}, A_i v_{in} \rangle + \frac{1}{r_{n}} \langle y - v_{in}, Jv_{in} - Jx_n \rangle \geq 0, \quad \forall y \in C, \quad (28)
\]
\[
\langle y - v_{in+1}, A_i v_{in+1} \rangle + \frac{1}{r_{n+1}} \langle y - v_{in+1}, Jv_{in+1} - Jx_{n+1} \rangle \geq 0, \\
\forall y \in C. \quad (29)
\]

Putting $y := v_{in+1}$ in (28) and letting $y := v_{in}$ in (29), we have that
\[
\langle v_{in+1} - v_{in}, A_i v_{in+1} \rangle + \frac{1}{r_{n+1}} \langle v_{in+1} - v_{in}, Jv_{in+1} - Jx_{n+1} \rangle \geq 0, \quad (30)
\]
\[
\langle v_{in} - v_{in+1}, A_i v_{in+1} \rangle + \frac{1}{r_{n+1}} \langle v_{in} - v_{in+1}, Jv_{in+1} - Jx_{n+1} \rangle \geq 0.
\]

Adding (30), we have that
\[
\langle v_{in+1} - v_{in}, A_i v_{in+1} \rangle \\
+ \left( v_{in+1} - v_{in} \right) \left( \frac{Jv_{in} - Jx_n}{r_{n}} - \frac{Jv_{in+1} - Jx_{n+1}}{r_{n+1}} \right) \geq 0. \quad (31)
\]

Since $A_i, i \in \{1, 2, \ldots, m\}$ are monotone mappings, which implies that
\[
\left( v_{in+1} - v_{in} \right) \left( \frac{Jv_{in} - Jx_n}{r_{n}} - \frac{Jv_{in+1} - Jx_{n+1}}{r_{n+1}} \right) \geq 0, \quad (32)
\]

therefore we have that
\[
\left( v_{in+1} - v_{in} \right) \frac{r_n}{r_{n+1}} \left( Jv_{in+1} - Jx_{n+1} \right) + Jv_{in+1} - Jv_{in} \right) \geq 0.
\]
(33)

That is,
\[
\frac{r_n}{r_{n+1}} \left( Jv_{in+1} - Jx_{n+1} \right) \geq 0.
\]

Without loss of generality, let $b$ be a real number such that $r_n > b > 0$; then we have that
\[
\|v_{in+1} - v_{in}\| \leq \|x_{n+1} - x_n\| + \frac{1 - r_n}{r_{n+1}} \|v_{in+1} - x_{n+1}\| \\
\leq \|x_{n+1} - x_n\| + \frac{1}{b} \left( \frac{r_{n+1}}{r_n} - r_n \right) K, \quad (35)
\]

where $K = \sup \|v_{in+1} - x_{n+1}\|$. Then, we have from (35) and (27) that
\[
\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \frac{1 - \lambda_{n+1}}{b} \left( \frac{r_{n+1}}{r_n} - r_n \right) K \\
+ |\lambda_{n+1} - \lambda_n| \|x_n - \sum_{i=1}^{m} \mu_i F_{fr_{n}} x_n\|. \quad (36)
\]

On the other hand, let $u_{in} = T_{fr_{n}} y_n, u_{in+1} = T_{fr_{n}} y_{n+1}$; we have that
\[
\langle y - u_{in}, T_{fr_{n}} y_n \rangle - \frac{1}{r_{n}} \langle y - u_{in}, J(y_{in+1} - Jy_{n}) \rangle \leq 0, \quad (37)
\]
\[
\forall y \in C.
\]

\[
\langle y - u_{in+1}, T_{fr_{n+1}} y_{n+1} \rangle - \frac{1}{r_{n+1}} \langle y - u_{in+1}, (1 + r_{n+1}) J(y_{in+1} - Jy_{n}) \rangle \leq 0, \quad (38)
\]
\[
\forall y \in C.
\]
Let \( y := u_{i,n+1} \) in (37), and let \( y := u_n \) in (38); we have that
\[
\begin{align*}
\langle u_{i,n+1} - u_n, T_{i} u_n \rangle - \frac{1}{r_n} (u_{i,n+1} - u_n, (1 + r_n) J u_n - J y_n) \leq 0, \\
\langle u_n - u_{i,n+1}, T_{i} u_{i,n+1} \rangle - \frac{1}{r_{n+1}} (u_n - u_{i,n+1}, (1 + r_{n+1}) J u_{i,n+1} - J y_{i,n+1}) \leq 0.
\end{align*}
\]  
Adding (39) and because \( T_{i} \), \( i \in \{1, 2, \ldots, m\} \) is pseudononexpansive, we have that
\[
\begin{align*}
\langle u_{i,n+1} - u_n, (u_{i,n+1} - u_n) \rangle + \frac{1}{r_{n+1}} (u_{i,n+1} - u_n, (1 + r_{n+1}) J u_{i,n+1} - J y_{i,n+1}) \geq 0.
\end{align*}
\]  
Therefore we have
\[
\begin{align*}
\langle u_{i,n+1} - u_n, J u_n - J y_n - \frac{r_n (J u_{i,n+1} - J y_{i,n+1})}{r_{n+1}} \rangle \geq 0.
\end{align*}
\]  
Hence we have that
\[
\begin{align*}
\|u_{i,n+1} - u_n\| \leq \|y_{i,n+1} - y_n\| + \frac{1}{b} |r_{n+1} - r_n| M,
\end{align*}
\]  
where \( M = \sup \{\|u_{i,n} - y_n\|\} \).

Let \( x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n \). Hence we have that
\[
\begin{align*}
z_{n+1} - x_n & = \frac{\alpha_n r_{n+1}}{1 - \beta_n} (f(x_{n+1}) - f(x_n)) \\
& + \left( \frac{\alpha_n r_{n+1}}{1 - \beta_n} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) \\
& + \frac{\gamma_{n+1}}{1 - \beta_n} (u_{i,n+1} - u_n) \\
& + \left( \frac{\gamma_{n+1}}{1 - \beta_n} - \frac{\gamma_n}{1 - \beta_n} \right) u_n.
\end{align*}
\]  
Hence we have from (43), (42), and (36) that
\[
\begin{align*}
\|z_{n+1} - x_n\| - \|x_{n+1} - x_n\| & \leq \left( \frac{\rho - 1}{1 - \beta_n} \right) \|x_{n+1} - x_n\| + \left| \frac{\alpha_n r_{n+1}}{1 - \beta_n} - \frac{\alpha_n}{1 - \beta_n} \right| \\
& \times (\|f(x_n)\| + \|u_n\|) + \frac{\gamma_{n+1}}{1 - \beta_n} |r_{n+1} - r_n| \frac{b}{b} \\
& \times ((1 - \lambda_{n+1}) M) \\
& \times \|x_n - \sum_{i=1}^{m} \mu_i F_{x_n} x_n\|.
\end{align*}
\]  
Noticing the conditions (ii) and (iv), we have that
\[
\begin{align*}
\limsup_{n \to \infty} \left( \|z_{n+1} - x_n\| - \|x_{n+1} - x_n\| \right) = 0.
\end{align*}
\]  
Hence we have from Lemma 4 that
\[
\begin{align*}
\limsup_{n \to \infty} \|z_n - x_n\| = 0.
\end{align*}
\]  
Therefore we have that
\[
\begin{align*}
\|x_{n+1} - x_n\| = \left| 1 - \beta_n \right| \|z_n - x_n\| \to 0.
\end{align*}
\]  
Hence we have from (35), (36), and (42) that
\[
\begin{align*}
\|y_{n+1} - y_n\| \to 0, \\
\|u_{i,n+1} - u_n\| \to 0, \\
\|v_{i,n+1} - v_n\| \to 0.
\end{align*}
\]  
In addition, since \( x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i u_{i,n} \), \( y_n = \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{m} \mu_i v_{i,n} \), for all \( p \in F \), we have from the monotonicity of \( A_i \), the non-expansivity of \( T_{i,n} \), and the convexity of \( \|\cdot\| \) that
\[
\begin{align*}
\|x_{n+1} - p\|^2 & = \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i u_{i,n} - p \right\|^2 \\
& \leq \|\alpha_n (f(x_n) - p) + \beta_n (x_n - p)\|^2 + \gamma_n \sum_{i=1}^{m} \|\sigma_i u_{i,n} - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + \gamma_n \|x_n - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + \gamma_n \|x_n - p\|^2 + (1 - \lambda_n) \gamma_n \|v_n - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + (\beta_n + \gamma_n) \|x_n - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + (\beta_n + \gamma_n) \|x_n - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + \|x_n - p\|^2 - (1 - \lambda_n) \gamma_n \|x_n - v_n\|^2.
\end{align*}
\]  
So we have that
\[
\begin{align*}
(1 - \lambda_n) \gamma_n \|x_n - v_n\|^2 & \leq \|\alpha_n f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
& \leq \|\alpha_n f(x_n) - p\|^2 + \|x_n - x_{n+1}\| \times (\|x_n - p\| + \|x_{n+1} - p\|).
\end{align*}
\]  
Since \( \alpha_n \to 0 \), so we have from (47) that
\[
\begin{align*}
\|x_n - v_n\| \to 0.
\end{align*}
\]  
In a similar way, we have that
\[
\begin{align*}
\|x_n - u_{i,n}\| \to 0.
\end{align*}
\]
Consequently, we have that
\[
\|y_n - x_n\| = \|1 - \lambda_n\| \|x_n - y_n\| \to 0, \\
\|y_n - u_m\| \leq \|y_n - x_n\| + \|x_n - u_m\| \to 0.
\]
(53)

Since $E$ is a reflexive real Banach space and the sequence $\{x_n\}$ is bounded, so there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $w \in C$ such that $x_{n_k} \to w$. And because $x_n \to y_n$, $n \to \infty$, therefore $v_{n_k} \to w$. Next we show that $w \in F$.

Because $v_{n_k} = F_{ir_n}x_n$, by the definition of mapping $F_{ir_n}$, we have that
\[
\langle y - v_{n_k}, A_i v_{n_k} \rangle + \frac{1}{r_n} \langle y - v_{n_k}, Jv_{n_k} - Jx_{n_k} \rangle \geq 0, \quad \forall y \in C,
\]
\[
\langle y - v_{n_k}, A_i v_{n_k} \rangle + \langle y - v_{n_k}, Jv_{n_k} - Jx_{n_k} \rangle \geq 0, \quad \forall y \in C.
\]
(54)

Let $v_t = tv + (1 - t)w, t \in [0,1]$, for all $v \in C$; we have that
\[
\langle v_t - v_{n_k}, A_i v_t \rangle \geq \langle v_t - v_{n_k}, A_i v_{n_k} \rangle - \langle v_t - v_{n_k}, Jv_{n_k} - Jx_{n_k} \rangle
\]
\[
\quad = \langle v_t - v_{n_k}, A_i v_t - A_i v_{n_k} \rangle
\]
\[
\quad - \langle v_t - v_{n_k}, Jv_{n_k} - Jx_{n_k} \rangle.
\]
(55)

Because $x_{n_k} - v_{n_k} \to 0$, so $(Jv_{n_k} - Jx_{n_k})/r_n \to 0$, and because $A_i$ are monotone, we have that
\[
0 \leq \lim_{k \to \infty} \langle v_t - v_{n_k}, A_i v_{n_k} \rangle = \langle v_t - w, A_i v_t \rangle.
\]
(56)

Consequently we have that
\[
\langle v - w, A_i v_t \rangle \geq 0.
\]
(57)

If $t \to 0$, by the continuity of $A_i$, we have that $\langle v - w, A_i w \rangle \geq 0$; that is, $w \in VI(C, A_i)$, and then $w \in F_i$. Similarly, because $u_m = T_{ir_n}y_n$, by the definition of mapping $T_{ir_n}$, we have that
\[
\langle y - u_m, T_{ir_n}u_m \rangle - \frac{1}{r_n} \langle y - u_m, (1 + r_n) Ju_m - Jy_n \rangle \leq 0, \quad \forall y \in C,
\]
\[
\langle y - u_{n_k}, T_{ir_n}u_{n_k} \rangle - \frac{1}{r_n} \langle y - u_{n_k}, (1 + r_n) Ju_{n_k} - Jy_{n_k} \rangle \leq 0, \quad \forall y \in C.
\]
(58)

Let $v_t = tv + (1 - t)w, t \in [0,1]$, for all $v \in C$. Because $\{T_i\}, i \in \{1,2,\ldots,m\}$ are pseudocontractive mappings, we have that
\[
\langle u_{n_k} - v_t, T_i v_t \rangle
\]
\[
\geq \langle u_{n_k} - v_t, T_i v_t \rangle + \langle v_t - u_{n_k}, T_i u_{n_k} \rangle
\]
\[
- \frac{1}{r_n} \langle v_t - u_{n_k}, (1 + r_n) Ju_{n_k} - Jy_{n_k} \rangle
\]
\[
= \langle v_t - u_{n_k}, T_i u_{n_k} - T_i v_t \rangle
\]
\[
- \langle v_t - u_{n_k}, Ju_{n_k} \rangle
\]
\[
- \langle v_t - u_{n_k}, Ju_{n_k} \rangle
\]
(59)

Because $y_{n_k} - u_{n_k} \to 0$, so $Ju_{n_k} - Jy_{n_k} \to 0$; we have that
\[
\lim_{k \to \infty} \langle u_{n_k} - v_t, T_i v_t \rangle \geq \lim_{k \to \infty} \langle u_{n_k} - v_t, Jv_t \rangle.
\]
(60)

Consequently we have that
\[
\langle w - v_t, T_i v_t \rangle \geq \langle w - v_t, Jv_t \rangle,
\]
\[
\langle v - w, T_i v_t \rangle \leq \langle v - w, Jv_t \rangle.
\]
(61)

If $t \to 0$, by the continuity of $T_i$, we have that $\langle v - w, T_i w \rangle \geq 0$, for all $v \in C$; we conclude that $w = T_i w$; that is, $w \in F(T_i)$, and then $w \in F_i$. Consequently $w \in F = F_1 \cap F_2$.

Because $\bar{x} = \Pi_{F} f(x)$, we have from Lemma 5 that
\[
\lim_{n \to \infty} \sup \langle f(x) - \bar{x}, j(x_{n+1} - \bar{x}) \rangle = \langle f(x) - \bar{x}, j(w - \bar{x}) \rangle \leq 0.
\]
(62)

Next we show that $x_n \to \bar{x}$. Since $u_{m} = T_{ir_n}y_n$, from formula (23) and Lemma 6 we have that
\[
\|x_{n+1} - \bar{x}\| \leq \alpha_n\|f(x_n) - \bar{x}\| + \beta_n\|x_n - \bar{x}\| + \gamma_n\left(\sum_{i=1}^{m} \|s_iT_{ir_n}y_n - \bar{x}\|\right)^2
\]
\[
\leq \beta_n\|x_n - \bar{x}\| + \gamma_n\left(\sum_{i=1}^{m} \|s_iu_m - \bar{x}\|\right)^2
\]
\[
+ 2\alpha_n\|f(x_n) - \bar{x}\|, j(x_{n+1} - \bar{x})\)
\[ \langle f - I \rangle (\bar{x}), j (y - \bar{x}) \rangle \leq 0, \quad \forall y \in F. \]  

**Proof.** Putting \( \lambda_2 = 0 \) in Theorem 8, we can obtain the result. \( \square \)

If, in Theorems 8 and 9, we let \( f : = u \in C \) be a constant mapping, we have the following corollaries.

**Corollary 10.** Let \( C \) be a nonempty closed convex subset of a real reflexive Banach space \( E \) which has a uniformly Gateaux differentiable norm. Let \( T_i : C \to C, i = 1, 2, \ldots, m, \) be a finite family of continuous strictly pseudo-contractive mappings, and \( A_i : C \to E^*, i = 1, 2, \ldots, m, \) be a finite family of continuous monotone mappings such that \( F = F_1 \cap F_2 \neq 0, \) \( T_{ir}, \) and \( F_{ir} \) are defined as (19) and (20), respectively. Let \( x_n \) be a sequence generated by \( x_0 \in C \)

\[ y_n = \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{m} \mu_i T_{ir} x_n, \]  

\[ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i T_{ir} y_n, \]  

where \( \lambda_n \in [0, 1] \) and \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{\gamma_n\} \) are sequences of nonnegative real numbers in \([0, 1]\), and

\[ (i) \quad \alpha_n + \beta_n + \gamma_n = 1, \quad n \geq 0, \quad \sum_{i=1}^{m} e_i = 1, \quad e_i \geq 0; \]

\[ (ii) \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{i=1}^{m} \alpha_i = \infty; \]

\[ (iii) \quad 0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n < 1; \]

\[ (iv) \quad \limsup_{n \to \infty} \beta_n < 1, \quad \sum_{i=1}^{m} |r_{n+1} - r_n| < \infty. \]

Then the sequence \( x_n \) converges strongly to an element \( \bar{x} = \Pi_{F} f(x) \), and also \( \bar{x} \) is the unique solution of the variational inequality

\[ \langle (f - I) (\bar{x}), j(y - \bar{x}) \rangle \leq 0, \quad \forall y \in F. \]  

**Corollary 11.** Let \( C \) be a nonempty closed convex subset of a real reflexive Banach space \( E \) which has a uniformly Gateaux differentiable norm. Let \( T_i : C \to C, i = 1, 2, \ldots, m, \) be a finite family of continuous strictly pseudo-contractive mappings, and \( A_i : C \to E^*, i = 1, 2, \ldots, m, \) be a finite family of continuous monotone mappings such that \( F = F_1 \cap F_2 \neq 0, \) \( T_{ir}, \) and \( F_{ir} \) are defined as (19) and (20), respectively. Let \( x_n \) be a sequence generated by \( x_0 \in C \)

\[ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \sigma_i T_{ir} x_n, \]  

where \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{\gamma_n\} \) are sequences of nonnegative real numbers in \([0, 1]\), and

\[ (i) \quad \alpha_n + \beta_n + \gamma_n = 1, \quad n \geq 0, \quad \sum_{i=1}^{m} e_i = 1, \quad e_i > 0; \]

\[ (ii) \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{i=1}^{m} \alpha_i = \infty; \]

\[ (iii) \quad 0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n < 1; \]

\[ (iv) \quad \limsup_{n \to \infty} \beta_n < 1, \quad \sum_{i=1}^{m} |r_{n+1} - r_n| < \infty. \]
Then the sequence \( x_n \) converges strongly to \( x = \Pi_F u \), and also \( x \) is the unique solution of the variational inequality
\[
\langle u - x, f(y - x) \rangle \leq 0, \quad \forall y \in F.
\]

\[ (73) \]

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References

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