Research Article

Existence Results for a $p(x)$-Kirchhoff-Type Equation without Ambrosetti-Rabinowitz Condition

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Received 25 November 2012; Accepted 28 April 2013

Academic Editor: Jaime Munoz Rivera

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We consider the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equations without Ambrosetti-Rabinowitz condition. Using the Mountain Pass Lemma, the Fountain Theorem, and its dual, the existence of solutions and infinitely many solutions were obtained, respectively.

1. Introduction

The Kirchhoff equation

$$
\rho \frac{\partial^2 u}{\partial t^2} - \left( \rho_0 + \frac{E}{2L} \int_0^L |\partial u| \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0
$$

was introduced by Kirchhoff [1] in the study of oscillations of stretched strings and plates, where $\rho$, $\rho_0$, $h$, $E$, and $L$ are constants. The stationary analogue of the Kirchhoff equation, that is, (1), is, as follows:

$$
- \left( a + b \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = f(x,u).
$$

After the excellent work of Lions [2], problem (2) has received more attention; see [3–10] and references therein.

The $p(x)$-Laplace operator arises from various phenomena, for instance, the image restoration [11], the electro-rheological fluids [12], and the thermoconvective flows of non-Newtonian fluids [13,14]. The study of the $p(x)$-Laplace operator is based on the theory of the generalized Lebesgue space $L^{p(x)}(\Omega)$ and the Sobolev space $W^{m,p(x)}(\Omega)$, which we called variable exponent Lebesgue and Sobolev space. We refer the reader to [15–19] for an overview on the variable exponent Sobolev space, and to [20–29] for the study of the $p(x)$-Laplacian-type equations.

Recently, there has been an increasing interest in studying the Kirchhoff equation involving the $p(x)$-Laplace operator. Autuori et al. [30,31] have dealt with the nonstationary Kirchhoff-type equation involving the $p(x)$-Laplacian of the form

$$
u_{tt} - M \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \Delta_{p(x)} u
$$

$$
+ Q(t,x,u,u_t) + f(t,x,u) = 0,
$$

and

$$
u_{tt} - M \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \Delta_{p(x)} u
$$

$$
+ \mu |\nabla u|^{p(x)-2} u + Q(t,x,u,u_t) = f(t,x,u).
$$

In [32–35], applying variational method and Ambrosetti-Rabinowitz (AR) condition, Guowei Dai has studied the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equations with Dirichlet or Neumann boundary condition. In [36], by using ($S_\gamma$) mapping theory and the Mountain Pass Lemma, Fan has discussed the nonlocal $p(x)$-Laplacian Dirichlet problem with the nonvariational form

$$
-A(u) \Delta_{p(x)} u = B(u) f(x,u), \quad \text{in } \Omega,
$$

$$
u = 0, \quad \text{on } \partial \Omega.
$$
and the $p(x)$-Kirchhoff-type equation with the variational form
\[ -a \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \Delta_{p(x)} u = b \left( \int_{\Omega} F(x, u) \, dx \right) f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega, \]
under (AR) condition, where $A$, $B$ are two functionals defined on $W^{1,p(x)}(\Omega)$, and $F(x, t) = \int_0^t f(x, s) \, ds$.

Motivated by the above works, the purpose of this paper is to study the $p(x)$-Kirchhoff-type equation
\[ -\left( a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \Delta_{p(x)} u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega, \]
without (AR) condition, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $a, b$ are two positive constants, $\Delta_{p(x)} u = \text{div}(\nabla u |\nabla u|^{p(x)-2} \nabla u)$, $p \in C^0(\overline{\Omega})$ for some $\beta \in (0, 1)$, and
\[ 1 < p^- := \inf_{\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < +\infty. \]

By taking the famous Mountain Pass Lemma, the Fountain Theorem, and its dual, we obtain the existence of solutions and infinitely many solutions for the $p(x)$-Kirchhoff-type equation (6) under no (AR) condition.

2. Preliminary

We recall in this section some definitions and properties of variable exponent Lebesgue-Sobolev space. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by
\[ L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable, } \int_{\Omega} |u|^{p(x)} \, dx < \infty \right\}, \]
with the norm
\[ |u|_{L^{p(x)}} = \|u\|_{p(x)} = \inf \left\{ \sigma > 0 : \int_{\Omega} \frac{|u|}{\sigma}^{p(x)} \, dx \leq 1 \right\}. \]
The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by
\[ W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}, \]
with the norm
\[ \|u\|_{W^{1,p(x)}} = \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \]

Denote by $W^{1,p(x)}_0(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. $|\nabla u|_{p(x)}$ is an equivalent norm on $W^{1,p(x)}_0(\Omega)$. In this paper we use the notation $\|u\|_{p(x)}$ for $u \in W^{1,p(x)}_0(\Omega)$. Define
\[ p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases} \]
We refer the reader to [36–38] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

Proposition 1 (see [38]). Set $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx$. For any $u \in L^{p(x)}(\Omega)$, then the following are given:
\[ (1) |u|_{p(x)} = \lambda \Leftrightarrow p(u/\lambda) = 1 \text{ if } u \neq 0; \]
\[ (2) |u|_{p(x)} < 1 \Leftrightarrow p(u) < 1 \text{ if } u \neq 0; \]
\[ (3) \rho(u) \leq |u|_{p(x)} \Leftrightarrow |u|_{p(x)} \leq \rho(u) \text{ if } |u|_{p(x)} > 1; \]
\[ (4) \rho(u) \leq |u|_{p(x)} \Leftrightarrow |u|_{p(x)} \leq \rho(u) \text{ if } |u|_{p(x)} < 1; \]
\[ (5) \lim_{k \to +\infty} \rho(u_k) = 0 \Leftrightarrow \lim_{k \to +\infty} \rho(u_k) = 0; \]
\[ (6) \lim_{k \to +\infty} |u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k \to +\infty} \rho(u_k) = +\infty. \]

3. Positive Energy Solution

In this section we discuss the existence of weak solutions of (6). For simplicity we write $X = W^{1,p(x)}_0(\Omega)$.

First, we state the assumptions on $f$ as follows.
\[ (f_0) \text{ Let } f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ be a continuous function, and there exist positive constants } c_1, c_2 \text{ such that} \]
\[ |f(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1}, \]
where $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.
\[ (f_0') \text{ Let } f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ be a continuous function, and there exist positive constants } c_1, c_2 \text{ such that} \]
\[ |f(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1}, \]
where $\alpha \in C(\overline{\Omega})$ and $p^* \leq \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$; $tf(x, t) \geq 0$ for all $t > 0$.
\[ (f_1) \text{ Let } \lim_{t \to +\infty} F(x, t)/|t|^{2p^*} = +\infty, \text{ uniformly for } x \in \overline{\Omega}, \text{ where } F(x, t) = \int_0^t f(x, s) \, ds. \]
\[ (f_2) \text{ There exists } \theta \geq 1 \text{ such that } \theta G(x, st) \geq G(x, st) \text{ for } (x, t) \in \Omega \times \mathbb{R} \text{ and } s \in [0, 1], \text{ where} \]
\[ G(x, t) = tf(x, t) - 2p^*F(x, t). \]
\[ (f_3) \text{ Let } \lim_{t \to 0} F(x, t)/|t|^{p^*} = 0, \text{ uniformly on } x \in \Omega. \]
\[ (f_3') \text{ There exists } \delta > 0, \text{ such that } F(x, t) \leq 0 \text{ for } x \in \overline{\Omega}, |t| < \delta. \]
(17) Let \( f(x, t) = -f(x, -t) \) for \( x\in\Omega \) and \( t\in \mathbb{R} \).

(f5) Let \( \lim_{t\to 0}(F(x, t)/|t|^p) = 0 \), uniformly on \( x \in \overline{\Omega} \), where \( q \in C(\overline{\Omega}) \) satisfies \( 1 < q(x) < p(x) \) for \( x \in \overline{\Omega} \).

Remark 2. Condition (f5) was first introduced by Jeanjean [39] for the case \( p(x) = 2 \). Let \( f(x, t) = 2p^1\frac{t}{|t|^p} \ln|t| \), then

\[
F(x, t) = |t|^p \ln|t| - \frac{1}{2p^1} |t|^2p^1, \quad G(x, t) = |t|^2p^1.
\]

(16) It is easy to see that the function \( f \) does not satisfy (AR) condition, but it satisfies \((f_1)-(f_5)\) and \((f_5')\).

Proposition 4 (see [37]). Assume that \((f_5)\) hold and let \( u_0 \in W_0^{1,p}(\Omega) \) be a local minimizer (resp., a strictly local minimizer) of \( I \) in the \( C^1(\overline{\Omega}) \) topology. Then \( u_0 \) is a local minimizer (resp., a strictly local minimizer) of \( I \) in the \( W_0^{1,p}(\Omega) \) topology.

Definition 5. We say that \( u \in X \) is a weak solution of (6), if

\[
(a + b \int \frac{1}{p(x)}|\nabla u|^{p(x)} \ dx) \int \frac{1}{p(x)} |\nabla u|^{p(x)} \ dx, \quad \Phi(u) = \int F(x, u) \ dx.
\]

(18) \( I(u) = J(u) - \Phi(u) \), where

Then \( I \in C^1(X, \mathbb{R}) \).

Proposition 3 (see [38]). Assume that \((f_5)\) hold, then the functional \( J : X \to \mathbb{R} \) is sequentially weakly lower semicontinuous, \( \Phi : X \to \mathbb{R} \) is sequentially weakly continuous, and \( I \) is sequentially weakly lower semicontinuous.

Proposition 4 (see [37]). Assume that \((f_5)\) hold, and let \( u_0 \in W_0^{1,p}(\Omega) \) be a local minimizer (resp., a strictly local minimizer) of \( I \) in the \( C^1(\overline{\Omega}) \) topology. Then \( u_0 \) is a local minimizer (resp., a strictly local minimizer) of \( I \) in the \( W_0^{1,p}(\Omega) \) topology.

Definition 6. Let \( X \) be a Banach space and \( I \in C^1(X, \mathbb{R}) \). Given \( c \in \mathbb{R} \), we say that \( I \) satisfies the Cerami \( c \) condition (we denote by \((C)_c\) the condition), if

(i) any bounded sequence \( \{u_n\} \subset X \) such that \( I(u_n) \to c \) and \( I'(u_n) \to 0 \) has a convergent subsequence;

(ii) there exist constants \( \delta, R, \beta > 0 \) such that

\[
\|u\| I'(u) \geq \beta, \quad \forall u \in I^{-1} [c - \delta, c + \delta], \quad \|u\| \geq R.
\]

If \( I \in C^1(X, \mathbb{R}) \) satisfies \((C)_c\) condition for every \( c \in \mathbb{R} \), then we say that \( I \) satisfies \((C)\) condition.

Remark 7. Although \((PS)\) condition is stronger than \((C)\) condition, the Deformation Theorem is still valid under \((C)\) condition; we see that the Mountain Pass Lemma, the Fountain Theorem, and its dual are true under \((C)\) condition.

Lemma 8. Assume that conditions \((f_5)-(f_5')\) hold. Then \( I \) satisfies \((C)\) condition.

Proof. From [36, Proposition 3.1], \( I \) satisfies \((i)\) of \((C)\) condition.

Now we check that \( I \) satisfies \((ii)\) of \((C)\) condition. Arguing by contradiction, we may assume that, for some \( c \in \mathbb{R} \),

\[
I(u_n) \to c, \quad \|u_n\| \to \infty, \quad \|u_n\| I'(u_n) \to 0. \tag{20}
\]

Then we have

\[
\lim_{n \to \infty} \left\{ a \int \frac{1}{p(x)} |\nabla u|^{p(x)} \ dx + b \right\} = c.
\]

(21) \( I(u_n) \to c \), \( \|u_n\| \to \infty \), \( \|u_n\| I'(u_n) \to 0 \).

Let \( v_n = u_n/\|u_n\| \), then up to a subsequence we may assume that

\[
\begin{align*}
v_n & \to v \quad \text{in } X, \\
v_n & \to v \quad \text{in } L^{a(x)}(\Omega), \\
v_n(x) & \to v(x) \quad \text{a.e. } x \in \Omega.
\end{align*}
\]

(22) \( v_n \to v \) in \( X \), \( v_n \to v \) in \( L^{a(x)}(\Omega) \), \( v_n(x) \to v(x) \) a.e. \( x \in \Omega \).

If \( v = 0 \), inspired by [13, 14], then we define

\[
I(u_n) = \max_{t \in [0,1]} I(tu_n). \tag{23}
\]

(23) \( I(u_n) = \max_{t \in [0,1]} I(tu_n) \).

For any \( m > 1/2p^+ \), let \( w_n = (2mp^+)^{1/p^} v_n \). Since \( w_n \to 0 \) in \( L^{a(x)}(\Omega) \) and

\[
|F(x, t)| \leq c_9 + c_9 |t|^{a(x)}, \tag{24}
\]

(24) \( |F(x, t)| \leq c_9 + c_9 |t|^{a(x)} \),

by the continuity of \( F(x, \cdot) \), \( F(\cdot, w_n) \to 0 \) in \( L^1(\Omega) \), thus,

\[
\lim_{n \to 0} \int |F(\cdot, w_n)| \ dx = 0. \tag{25}
\]
Then for $n$ large enough, $(2mp^+)\frac{1}{p^+} \|u_n\| \in (0, 1)$ and

$$I(t_n u_n) \geq I(u_n)$$

$$= a \int_\Omega \frac{1}{p(x)} |\nabla u_n|^p(x) dx$$

$$+ b \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^p(x) dx \right)^2 - \int_\Omega F(x, u_n) dx$$

$$= a \int_\Omega \frac{1}{p(x)} \left((2mp^+)\frac{1}{p^+} |\nabla u_n|^p(x) \right) dx$$

$$+ b \left( \int_\Omega \frac{1}{p(x)} \left((2mp^+)\frac{1}{p^+} |\nabla u_n|^p(x) \right) dx \right)^2$$

$$- \int_\Omega F(x, u_n) dx$$

$$\geq \frac{2ma}{p^+} \int_\Omega |\nabla u_n|^p(x) dx$$

$$+ \frac{2mb}{(p^+)^2} \left( \int_\Omega |\nabla u_n|^p(x) dx \right)^2 - \int_\Omega F(x, u_n) dx$$

$$\geq \frac{2ma}{p^+} + \frac{2mb}{(p^+)^2} - \int_\Omega F(x, u_n) dx.$$

(26)

That is, $I(t_n u_n) \to \infty$. From $I(0) = 0$ and $I(u_n) \to c$, we know that $t_n \in (0, 1)$ and

$$a \int_\Omega |\nabla_t u_n|^p(x) dx$$

$$+ b \left( \int_\Omega \frac{1}{p(x)} |\nabla_t u_n|^p(x) dx \right) \int_\Omega |\nabla u_n|^p(x) dx$$

$$- \int_\Omega f(x, t_n u_n) u_n dx$$

$$= \langle I'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt} I(t_n u_n) = 0.$$

(27)

Therefore, from $(f_2)$, we have

$$a \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{2p^+} \right)|\nabla u_n|^p(x) dx$$

$$+ b \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^p(x) dx \right) + \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{p^+} \right)|\nabla u_n|^p(x) dx$$

$$+ \int_\Omega G(x, u_n) dx$$

$$\geq a \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{2p^+} \right)|\nabla u_n|^p(x) dx$$

$$+ \frac{b}{2} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^p(x) dx$$

$$\times \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{p^+} \right)|\nabla u_n|^p(x) dx$$

$$+ \frac{1}{2p^+} \int_\Omega G(x, u_n) dx$$

(28)

This contradicts (21).

If $v \neq 0$, from (20), when $\|u_n\| \geq 1$,

$$\frac{a}{p^+} \|u_n\|^{p^+} + \frac{b}{2(p^+)^2} \|u_n\|^{2p^+} - (c + o(1)) \geq \int_\Omega F(x, u_n) dx.$$

(29)

Then from $(f_1)$ we have

$$\frac{a}{p^+} \frac{1}{\|u_n\|^{p^+}} + \frac{b}{2(p^+)^2} \|u_n\|^{2p^+} - (c + o(1)) \geq \int_\Omega F(x, u_n) dx$$

$$\geq \int_\Omega \frac{F(x, u_n)}{\|u_n\|^{2p^+}} dx$$

$$= \left( \int_{u_n \neq 0} + \int_{u_n = 0} \right) \frac{F(x, u_n)}{|u_n|^{2p^+}} dx$$

$$= \int_{u_n \neq 0} \frac{F(x, u_n)}{|u_n|^{2p^+}} dx.$$

(30)

For $x \in \Theta := \{ x \in \Omega : v(x) \neq 0 \}, |u_n(x)| \to +\infty$. By $(f_1)$ we have

$$\frac{F(x, u_n)}{|u_n|^{p^+}} |v_n|^{p^+} \to +\infty.$$

(31)
Note that the Lebesgue measure of $\Theta$ is positive; using the Fatou Lemma, we have
\[ \int_{r_n > 0} \frac{F(x,u_n)}{|u_n|^{\alpha(p)}} |\nabla u_n|^{\beta} \, dx \to +\infty. \] (32)
This contradicts (30).

The technique used in this lemma was first introduced by [39, 40]. \qed

**Theorem 9.** Assume that conditions (f0)—(f2) and (f3) (or (f3')) hold. Then (6) has a nontrivial solution with positive energy.

**Proof.** From Lemma 8, $I$ satisfies (C) condition. Let us show that the functional $I$ has a Mountain-Pass-type geometry.

Note that $I(0) = 0$. By (f2), there exists $\delta > 0$, and for any $u \in X$ with $|u|_{L^p(\Omega)} < \delta$, $I(u) = a \left( \int_{\Omega} \frac{1}{p(x)} |V u|^{p(x)} \, dx \right)$
\[ + \frac{b}{2} \left( \int_{\Omega} \frac{1}{p(x)} |V u|^{p(x)} \, dx \right)^2 - \int_{\Omega} F(x,u) \, dx \geq \frac{a}{p} |u|^{p^*} + \frac{b}{(p^*)^2} |u|^{2p^*} - \int_{\Omega} F(x,u) \, dx > 0. \] (33)

This shows that $0$ is a strictly local minimizer of $I$ in the $C(\overline{\Omega})$ topology, and hence $0$ is a strictly local minimizer of $I$ in the $C^1(\overline{\Omega})$ topology by [37, Theorem 1.1]. $0$ is a strictly local minimizer of $I$ in the $W^{1,p(x)}(\Omega)$ topology. Thus there exists $r > 0$ such that $I(u) > 0$ for every $u \in X \setminus \{0\}$ with $|u| \leq r$.

We claim that \(\inf_{\|u\| = r} I(u) > 0\). To prove this claim, arguing by contradiction, assume that there exists a sequence $\{u_n\} \subset X$ with $\|u_n\| = r$ such that $I(u_n) \to 0$ as $n \to \infty$. We may assume that $u_n \rightharpoonup u_0$ in $X$. Since $I$ is sequentially weakly semi-continuous, we have that $I(u_0) = 0$, and hence $u_0 = 0$. Since $\Phi$ is sequentially weakly continuous, then we have that $\Phi(u_n) \to \Phi(0) = 0$, and hence $I(u_n) = I(u_0) + \Phi(u_n) \to 0$. It follows from this that $u_n \to 0$ in $X$ which contradicts with $\|u_n\| = r$.

Let $y \in X$ with $y > 0$ in $\Omega$ and $\|y\| = 1$. By (f0) and (f2), for $s \geq 1$ we have
\[ I(sy) = a \left( \int_{\Omega} \frac{1}{p(x)} |V y|^{p(x)} \, dx \right) \]
\[ + b \left( \int_{\Omega} \frac{1}{p(x)} |V y|^{p(x)} \, dx \right)^2 - \int_{\Omega} F(x,y) \, dx \leq \frac{a}{p} s^{p^*} + \frac{b}{(p^*)^2} s^{2p^*} - c_1 s^{p^*} \int_{\Omega} |y|^{2p^*} \, dx + c_2 \to -\infty \text{ as } s \to +\infty. \] (34)

We set $e = sy$. Then for $s$ large, we obtain
\[ \|e\| > r, \quad I(e) < 0. \] (35)

Hence by the famous Mountain Pass Lemma, problem (6) has a nontrivial weak solution with positive energy. \qed

**4. Infinitely Many Solutions**

Since $X$ is a reflexive and separable Banach space, then there exists $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that
\[ X = \text{span} \{e_j : j = 1, 2, \ldots\}, \]
\[ X^* = \text{span} \{e_j^* : j = 1, 2, \ldots\}, \] (36)
\[ \langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \]

For convenience, we write $X_j = \text{span}[e_j], Y_k = \Phi^k_{j=1} X_j, Z_k = \Phi^{\infty}_{j=k} X_j$.

**Lemma 10** (see [21]). If $\alpha \in C(\overline{\Omega}), 1 < \alpha(x) < p^*$ for any $x \in \overline{\Omega}$, denote
\[ \beta_k = \sup \{ |u|_{\alpha(x)} : |u| = 1, u \in Z_k \}. \] (37)

Then \(\lim_{k \to +\infty} \beta_k = 0\).

**Proposition 11** (Fountain Theorem). Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k \in \mathbb{N}$, there exists $\rho_k > \rho_k > 0$ such that
\[ (A_1) \quad a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0, \]
\[ (A_2) \quad b_k = \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \to +\infty \text{ as } k \to \infty, \]
\[ (A_3) \quad I \text{ satisfies (C)}_c \text{ condition for every } c > 0, \text{ then } I \text{ has an unbounded sequence of critical values.} \]

**Proposition 12** (Dual Fountain Theorem). Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k > k_0$, there exists $\rho_k > \rho_k > 0$ such that
\[ (B_1) \quad a_k = \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \geq 0, \]
\[ (B_2) \quad b_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) < 0, \]
\[ (B_3) \quad d_k = \inf_{u \in Z_k, \|u\| < \rho_k} I(u) \to 0 \text{ as } k \to \infty, \]
\[ (B_4) \quad I \text{ satisfies (C)}_c^* \text{ condition for every } c \in [d_k, \rho_k], \text{ then } I \text{ has a sequence of negative critical values converging to } 0. \]

**Theorem 13.** Assume that the conditions (f0'), (f1)–(f3) hold. Then (6) has infinitely many solutions $\{u_k\}$ such that $I(u_k) \to \infty$ as $k \to \infty$.

**Proof.** By conditions (f0'), (f1), and (f3), for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that
\[ F(x,u) \geq C_\varepsilon |u|^{2p^*} - \varepsilon |u|^p, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \] (38)
For $u \in Y_k$, when $\|u\| > 1$,

\[
I(u) = a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \\
+ \frac{b}{2} \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^2 - \int_\Omega F(x,u) \, dx \\
\leq \frac{a}{p} \|u\|^{p^*} + \frac{b}{2(p^*)^2} \|u\|^{2p^*} \\
- C_k |u|^{2p^*}_{2p^*} + \varepsilon |u|^{p^*}_{p^*} \to -\infty \quad \text{as } \|u\| \to +\infty.
\]

(39)

Then for some $\rho_k > 0$ large enough,

\[
a_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0.
\]

(40)

On the other hand, by $(f_0')$ and $(f_3)$, there exists $C_\varepsilon > 0$ such that

\[
|F(x,u)| \leq \varepsilon |u|^{p^*} + C_\varepsilon |u|^{a(x)}, \quad \forall (x,u) \in \Omega \times \mathbb{R}.
\]

(41)

Let $\beta_k := \sup_{u \in Z_k, \|u\| = \rho_k} |u|_{a^*}$. From Lemma 10, $\beta_k \to 0$ as $k \to \infty$. For $u \in Z_k$, when $|u| \leq 1$ and $\varepsilon$ small enough,

\[
I(u) = a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \\
+ \frac{b}{2} \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^2 - \int_\Omega F(x,u) \, dx \\
\geq \frac{a}{p^*} \|u\|^{p^*} - \frac{b}{2(p^*)^2} \|u\|^{2p^*} - C_\varepsilon |u|^{a^*} - \varepsilon |u|^{p^*}_{p^*}.
\]

(42)

If we choose $r_k := (a/4c^p \beta_k^{a^*})^{1/(a-\rho)} \to \infty$ as $k \to \infty$, then, for $u \in Z_k$ with $\|u\| = r_k$,

\[
I(u) \geq \frac{a}{4p^*(4c^p \beta_k^{a^*})^{1/(a-\rho)}} := \tilde{b}_k,
\]

(43)

which implies that $b_k := \inf_{u \in Z_k, \|u\| = r_k} I(u) \geq \tilde{b}_k \to +\infty$ as $k \to +\infty$.

\[\square\]

**Theorem 14.** Assume that conditions $(f_0')$, $(f_1)$, $(f_2)$, $(f_3)$, and $(f_4')$ hold. Then (6) has infinitely many solutions $\{u_k\}$ such that $I(u_k) \to 0$ as $k \to \infty$.

\[\square\]

*Proof.* By conditions $(f_0')$, $(f_1)$, and $(f_3)$, for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that

\[
F(x,u) \geq C_\varepsilon |u|^{p^*} - \varepsilon |u|^{q^*}, \quad \forall (x,u) \in \Omega \times \mathbb{R}.
\]

(44)

For $u \in Y_k$, when $\|u\|$ is large enough,

\[
I(u) = a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \\
+ \frac{b}{2} \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^2 - \int_\Omega F(x,u) \, dx \\
\leq \frac{a}{p} \|u\|^{p^*} + \frac{b}{(p^*)^2} \|u\|^{2p^*} - C_\varepsilon |u|^{2p^*}_{2p^*} + \varepsilon |u|^{p^*}_{p^*} \to -\infty \quad \text{as } \|u\| \to +\infty.
\]

(45)

Then for some $r_k > 0$ large enough,

\[
b_k := \max_{u \in Y_k, \|u\| = r_k} I(u) < 0.
\]

(46)

On the other hand, by $(f_2)$, there exists $C_\varepsilon > 0$ such that

\[
|F(x,u)| \leq \varepsilon |u|^{q^*} + C_\varepsilon |u|^{a(x)}, \quad \forall (x,u) \in \Omega \times \mathbb{R}.
\]

(47)

Let $\beta_k := \sup_{u \in Z_k, \|u\| = r_k} |u|_{q^*}$, then $\beta_k \to 0$ as $k \to \infty$. For $u \in Z_k$, when $|u|$ and $\varepsilon$ small enough,

\[
I(u) = a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \\
+ \frac{b}{2} \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^2 - \int_\Omega F(x,u) \, dx \\
\geq \frac{a}{p^*} \|u\|^{p^*} - cC_\varepsilon |u|^{q^*} - ce |u|^{q^*}_{q^*}.
\]

(48)

If we choose $\rho_k := (4c^p \beta_k^{q^*}/a)^{1/(p^*-q^*)} \to 0$ as $k \to \infty$, then, for $u \in Z_k$ with $\|u\| = \rho_k$,

\[
I(u) \geq c\beta_k^{q^*} \left( 4c^p \beta_k^{q^*}/a \right)^{-q^*/(p^*-q^*)} := \tilde{a}_k,
\]

(49)

which implies that $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \geq \tilde{a}_k \to 0$ as $k \to +\infty$.

Furthermore, if $u \in Z_k$ with $\|u\| \leq \rho_k$, then

\[
I(u) \geq -c\beta_k^{q^*} \rho_k^{q^*} \to 0 \quad \text{as } k \to \infty,
\]

(50)

which implies that $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} I(u) \to 0$ as $k \to \infty$.

\[\square\]
Acknowledgment

This paper is supported by the National Natural Science Foundation of China (11126339, 11201008).

References


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