Research Article

Some Fixed Point Theorems for Nonlinear Set-Valued Contractive Mappings

Zeqing Liu,1 Zhihua Wu,1 Shin Min Kang,2 and Sunhong Lee2

1 Department of Mathematics, Liaoning Normal University, Dalian, Liaoning 116029, China
2 Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

Correspondence should be addressed to Sunhong Lee, sunhong@gnu.ac.kr

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Four fixed point theorems for nonlinear set-valued contractive mappings in complete metric spaces are proved. The results presented in this paper are extensions of a few well-known fixed point theorems. Two examples are also provided to illustrate our results.

1. Introduction and Preliminaries

The existence of fixed points for various set-valued contractive mappings had been researched by many authors under different conditions, see, for example, [1–9] and the references cited therein. In 1969, Nadler [7] proved a well-known fixed point theorem for the set-valued contraction mapping (1.1) below.

**Theorem 1.1 (see [7]).** Let $(X,d)$ be a complete metric space and $T : X \to \text{CB}(X)$ be a set-valued mapping such that

$$H(Tx,Ty) \leq rd(x,y), \quad \forall x,y \in X,$$

where $r \in (0,1)$ is a constant. Then $T$ has a fixed point.

In 1972, Reich [8] extended Nadler’s result and established an interesting fixed point theorem for the set-valued contraction mapping (1.2) below.

**Theorem 1.2 (see [8]).** Let $(X,d)$ be a complete metric space and $T : X \to C(X)$ satisfy that

$$H(Tx,Ty) \leq \varphi(d(x,y))d(x,y), \quad \forall x,y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a non-decreasing function with $\varphi(t) < t$ for all $t > 0$.
where

\[ \varphi : (0, +\infty) \to [0, 1) \text{ with } \limsup_{r \to t} \varphi(r) < 1, \quad \forall t \in (0, +\infty). \]  

(1.3)

Then \( T \) has a fixed point.

In [8] Reich posed the question whether Theorem 1.2 is also true for the set-valued contractive mapping \( T : X \to CB(X) \) with (1.2). The affirmative answer under the hypothesis of \( \limsup_{r \to t} \varphi(r) < 1 \), for all \( t \in [0, +\infty) \) was given by Mizoguchi and Takahashi in [6]. They deduced the following fixed point theorem which is a generalization of the Nadler fixed point theorem.

**Theorem 1.3** (see [6]). Let \((X, d)\) be a complete metric space and \( T : X \to CB(X) \) satisfy (1.2), where

\[ \varphi : (0, +\infty) \to [0, 1) \text{ with } \limsup_{r \to t} \varphi(r) < 1, \quad \forall t \in [0, +\infty). \]  

(1.4)

Then \( T \) has a fixed point.

**Remark 1.4.** It is clear that the mappings \( T \) in Theorems 1.1–1.3 are continuous on \( X \).

**Remark 1.5.** Each of Theorems 1.2 and 1.3 ensures that \( T \) has a fixed point \( a \in Ta \subseteq X \), which together with (1.2) implies that \( \varphi(0) = \varphi(d(a, a)) \), that is, \( \varphi \) is defined at 0. Thus the domain of \( \varphi \) in each of (1.3) and (1.4) should be \([0, +\infty)\) but not \((0, +\infty)\).

The aim of this paper is to present four fixed point theorems for some nonlinear set-valued contractive mappings. Our results extend, improve, and unify the corresponding results in [6–8]. Two nontrivial examples are given to show that our results are genuine generalizations or different from these results in [6–8].

Throughout this paper, we assume that \( \mathbb{R} = (-\infty, +\infty) \), \( \mathbb{R}^+ = [0, +\infty) \), \( \mathbb{N} \) and \( \mathbb{N}_0 \) denote the sets of all positive integers and nonnegative integers, respectively, and

\[ \Theta = \{ \theta : \theta : \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies } (a)-(d) \}, \]  

(1.5)

where

(a) \( \theta \) is nondecreasing on \( \mathbb{R}^+ \);

(b) \( \theta(t) > 0 \), for all \( t \in (0, +\infty) \);

(c) \( \theta \) is subadditive in \((0, +\infty)\), that is,

\[ \theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2), \quad \forall t_1, t_2 \in (0, +\infty); \]  

(1.6)

(d) \( \theta(\mathbb{R}^+) = \mathbb{R}^+ \).

Clearly (a)–(d) imply that

(e) \( \theta \) is strictly inverse on \( \mathbb{R}^+ \), that is, if there exist \( t, s \in \mathbb{R}^+ \) satisfying \( \theta(t) < \theta(s) \), then \( t < s \).
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Let \((X,d)\) be a metric space, \(CL(X)\), \(CB(X)\), and \(C(X)\) denote the families of all non-empty closed, all non-empty bounded closed, and all non-empty compact subsets of \(X\). For \(x \in X\) and \(A, B \in CL(X)\), put \(d(x, A) = \inf\{d(x, y) : y \in A\}\) and

\[
H(A, B) = \begin{cases} 
\max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}, & \text{if the maximum exists} \\
+\infty, & \text{otherwise}.
\end{cases}
\]  

(1.7)

Such a mapping \(H\) is called a generalized Hausdorff metric induced by \(d\) in \(CL(X)\). It is well known that \(H\) is a metric on \(CB(X)\). Let \(T : X \to CL(X)\) be a set-valued mapping, \(x_0 \in X\) and \(f : X \to \mathbb{R}^+\) be defined by

\[
f(x) = d(x, Tx), \quad \forall x \in X.
\]  

(1.8)

A sequence \(\{x_n\}_{n \in \mathbb{N}_0}\) is said to be an orbit of \(T\) if it satisfies that \(\{x_n\}_{n \in \mathbb{N}_0} \subseteq X\) and \(x_n \in T x_{n-1}\) for each \(n \in \mathbb{N}_0\). The function \(f : X \to \mathbb{R}^+\) is said to be \(T\)-orbitally lower semicontinuous at \(z \in X\) if for each orbit \(\{x_n\}_{n \in \mathbb{N}_0}\) of \(T\) with \(\lim_{n \to \infty} x_n = z\), we have that \(f(z) \leq \lim \inf_{n \to \infty} f(x_n)\).

2. Main Results

The following lemmas play important roles in this paper.

**Lemma 2.1.** Let \((X,d)\) be a metric space and \(B \in CL(X)\). Then for each \(x \in X\) and \(\varepsilon > 0\) there exists \(b \in B\) satisfying \(d(x, b) \leq d(x, B) + \varepsilon\).

**Proof.** Suppose that there exist \(x_0 \in X\) and \(\varepsilon_0 > 0\) such that

\[
d(x_0, b) > d(x_0, B) + \varepsilon_0, \quad \forall b \in B,
\]  

(2.1)

which yields that

\[
d(x_0, B) = \inf_{b \in B} d(x_0, b) \geq d(x_0, B) + \varepsilon_0 > d(x_0, B),
\]  

(2.2)

which is a contradiction. This completes the proof. \(\square\)

**Lemma 2.2.** Let \((X,d)\) be a metric space, \(B \in CL(X)\) and \(\theta \in \Theta\). Then for each \(x \in X\) and \(q > 1\) there exists \(b \in B\) such that

\[
\theta(d(x, b)) \leq q \theta(d(x, B)).
\]  

(2.3)

**Proof.** Let \(x \in X\) and \(q > 1\). Now we consider two possible cases as follows.

**Case 1.** Suppose that \(\theta(d(x, B)) = 0\). It follows from (b) and (d) that \(d(x, B) = 0\). Since \(B\) is a closed subset of \(X\), it follows that \(x \in B\). Put \(b = x\). Clearly (2.3) holds.
Case 2. Suppose that $\theta(d(x,B)) > 0$. Note that (b) and (d) mean that

$$\theta(d(x,B)) \in \mathbb{R}^+ \setminus \{0\} = \theta(\mathbb{R}^+ \setminus \{0\}).$$

(2.4)

Choose $p \in \theta^{-1}((q-1)\theta(d(x,B)))$ and $\varepsilon = p/2 > 0$. Lemma 2.1 ensures that there exists $b \in B$ satisfying $d(x,b) \leq d(x,B) + \varepsilon$, which together with (a) and (c) gives that

$$\theta(d(x,b)) \leq \theta(d(x,B) + \varepsilon) \leq \theta(d(x,B)) + \theta(\varepsilon)$$

$$\leq \theta(d(x,B)) + \theta(\theta^{-1}((q-1)\theta(d(x,B)))) = q\theta(d(x,B)).$$

(2.5)

That is, (2.3) holds. This completes the proof. \(\square\)

Now we prove four fixed point theorems for the nonlinear set-valued contractive mappings (2.6), (2.25), (2.26), and (2.36) below in complete metric spaces.

**Theorem 2.3.** Let $(X,d)$ be a complete metric space and $T : X \to \text{CL}(X)$ satisfy that

$$\theta(d(y,Ty)) \leq \varphi(d(x,y))\theta(d(x,y)), \quad \forall (x,y) \in X \times Tx,$$

(2.6)

where $\theta \in \Theta$ and

$$\varphi : \mathbb{R}^+ \to [0,1) \text{ with } \limsup_{r \to 1^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+.$$  

(2.7)

Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of $T$ and $z \in X$ such that $\lim_{n \to \infty} x_n = z$. Furthermore, $z \in X$ is fixed point of $T$ if and only if the function $f$ defined by (1.8) is $T$ orbitally lower semicontinuous at $z$.

**Proof.** Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.6), (2.7) and Lemma 2.2 that for $q_1 = 1/\max\{\sqrt[4]{\varphi(d(x_0,x_1))},1/2\} > 1$ there exists $x_2 \in Tx_1$ satisfying

$$\theta(d(x_1,x_2)) \leq \frac{\theta(d(x_1,Tx_1))}{\max\{\sqrt[4]{\varphi(d(x_0,x_1))},1/2\}} \leq \frac{\varphi(d(x_0,x_1))\theta(d(x_0,x_1))}{\max\{\sqrt[4]{\varphi(d(x_0,x_1))},1/2\}}$$

$$\leq \sqrt[4]{\varphi(d(x_0,x_1))}\theta(d(x_0,x_1)),$$

(2.8)

and for $q_2 = 1/\max\{\sqrt[4]{\varphi(d(x_1,x_2))},1/3\} > 1$ there exists $x_3 \in Tx_2$ satisfying

$$\theta(d(x_2,x_3)) \leq \frac{\theta(d(x_2,Tx_2))}{\max\{\sqrt[4]{\varphi(d(x_1,x_2))},1/3\}} \leq \frac{\varphi(d(x_1,x_2))\theta(d(x_1,x_2))}{\max\{\sqrt[4]{\varphi(d(x_1,x_2))},1/3\}}$$

$$\leq \sqrt[4]{\varphi(d(x_1,x_2))}\theta(d(x_1,x_2)).$$

(2.9)
Repeating the above argument we obtain a sequence \( \{x_n\}_{n \in \mathbb{N}_0} \subset X \) such that \( x_k \in Tx_{k-1} \) for \( 1 \leq k \leq n \) and for \( q_n = 1/\max \{ \sqrt{\varphi(d(x_{n-1},x_n))}, 1/(n+1) \} > 1 \), there exists \( x_{n+1} \in Tx_n \) satisfying

\[
\theta(d(x_n, x_{n+1})) \leq \frac{\theta(d(x_n, Tx_n))}{\max \{ \sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1) \}} \\
\leq \frac{\varphi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n))}{\max \{ \sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1) \}} \\
\leq \sqrt{\varphi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n))}, \quad \forall n \geq 1. \tag{2.10}
\]

Suppose that there exists some \( n_0 \in \mathbb{N}_0 \) satisfying \( x_{n_0} = x_{n_0+1} \in Tx_{n_0} \). It follows from (a), (b), and (2.10) that \( x_n = x_{n_0} \) for all \( n \geq n_0 + 1 \). It is clear the conclusion of Theorem 2.3 holds.

Suppose that \( x_{n+1} \in Tx_n \setminus \{x_n\} \) for any \( n \in \mathbb{N}_0 \). It follows that \( d(x_n, x_{n+1}) > 0 \) for each \( n \in \mathbb{N}_0 \). Note that (b), (2.7), and (2.10) give that \( \{\theta(d(x_n, x_{n+1}))\}_{n \in \mathbb{N}_0} \) is a positive and decreasing sequence. It follows from (e) that \( \{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0} \) is decreasing. Therefore, there exist constants \( p \) and \( q \) satisfying

\[
\lim_{n \to \infty} \theta(d(x_n, x_{n+1})) = p \geq 0, \quad \lim_{n \to \infty} d(x_n, x_{n+1}) = q \geq 0. \tag{2.11}
\]

Notice that (2.7) implies that there exists a constant \( r \) satisfying

\[
\limsup_{n \to \infty} \varphi(d(x_n, x_{n-1})) \leq \limsup_{t \to q^+} \varphi(t) = r \in [0, 1). \tag{2.12}
\]

Taking upper limits in (2.10) and by (2.11) and (2.12) we get that

\[
p \leq \sqrt{\limsup_{n \to \infty} \varphi(d(x_{n-1}, x_n))\limsup_{n \to \infty} \theta(d(x_{n-1}, x_n))} \leq \sqrt{rp}, \tag{2.13}
\]

which implies that \( p = 0 \).

Next we assert that \( q = 0 \). Since \( \{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0} \) is a decreasing sequence, it follows from (a) and (2.11) that

\[
0 \leq \theta(q) < \theta(d(x_n, x_{n+1})) \quad \text{as} \quad n \to \infty, \tag{2.14}
\]

that is, \( \theta(q) = 0 \), which together with (b) and (d) yields that \( q = 0 \).

Put \( c = (1 + r)/2 \). It follows from (2.12) that \( c \in (r, 1) \subset [0, 1) \), which gives that \( c^2 \in (r, 1) \). Notice that (2.11), (2.12), and \( q = 0 \) ensure that there exist \( \delta > 0 \) and \( N \in \mathbb{N} \) satisfying

\[
\varphi(t) < c^2, \quad \forall t \in (0, \delta), \quad d(x_n, x_{n+1}) < \delta, \quad \forall n \geq N, \tag{2.15}
\]
which implies that

\[ \varphi(d(x_n, x_{n+1})) < c^2, \quad \forall n \geq N. \] (2.16)

Note that (2.10) and (2.16) mean that

\[ \Theta(d(x_n, x_{n+1})) \leq \prod_{k=N}^{n-1} \varphi(d(x_k, x_{k+1})) \theta(d(x_N, x_{N+1})) \]

\[ \leq c^{n-N} \theta(d(x_N, x_{N+1})), \quad \forall n \geq N. \] (2.17)

Given \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} c^{n-N} \theta(d(x_N, x_{N+1})) = 0 \), it follows from (b) that there exists \( N_1 > N \) satisfying

\[ \frac{c^{n-N}}{1-c} \theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall n \geq N_1, \] (2.18)

which together with (2.17), (a), and (c) gives that

\[ \Theta(d(x_n, x_m)) \leq \Theta\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq \sum_{k=n}^{m-1} \Theta(d(x_k, x_{k+1})) \]

\[ \leq \sum_{k=n}^{m-1} c^{k-N} \theta(d(x_N, x_{N+1})) \]

\[ \leq \frac{c^{n-N}}{1-c} \theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall m > n \geq N_1. \] (2.19)

In view of (e) and (2.19), we deduce that \( d(x_n, x_m) < \varepsilon \), for all \( m > n \geq N_1 \), which means that \( \{x_n\}_{n \in \mathbb{N}_0} \) is a Cauchy sequence. Hence there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \) by completeness of \( X \).

Suppose that \( f \) is \( T \) orbitally lower semicontinuous at \( z \). Since \( \{x_n\}_{n \geq 0} \) is an orbit of \( T \) with \( \lim_{n \to \infty} x_n = z \), it follows that

\[ f(z) \leq \liminf_{n \to \infty} f(x_n). \] (2.20)

Using (2.6) and (2.7), we infer that

\[ \Theta(d(x_n, Tx_n)) \leq \varphi(d(x_{n-1}, x_n)) \Theta(d(x_{n-1}, x_n)) < \Theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \] (2.21)
Theorem 2.5. Let 

$$0 < d(x_\eta, Tx_\eta) < d(x_{\eta-1}, x_\eta) \to 0 \quad \text{as } n \to \infty,$$

(2.22)

that is, \( \lim_{n \to \infty} d(x_\eta, Tx_\eta) = 0 \), which together with (2.20) yields that

$$0 \leq d(z, Tz) = f(z) \leq \liminf_{n \to \infty} f(x_n) = \lim_{n \to \infty} d(x_n, Tx_n) = 0,$$

(2.23)

which gives that \( d(z, Tz) = 0 \), that is, \( z \in Tz \).

Conversely, suppose that \( z \in X \) is a fixed point of \( T \). Let \( \{y_n\}_{n \in \mathbb{N}} \subset X \) be an arbitrarily orbit of \( T \) with \( \lim_{n \to \infty} y_n = z \). It is clear that

$$f(z) = d(z, Tz) = 0 \leq \liminf_{n \to \infty} f(y_n),$$

(2.24)

which implies that \( f \) is \( T \) orbitally lower semicontinuous at \( z \). This completes the proof. \( \square \)

Notice that \( d(y, Ty) \leq H(Tx, Ty) \) for each \( y \in Tx \). In light of Theorem 2.3, we have

**Theorem 2.4.** Let \( (X, d) \) be a complete metric space and \( T : X \to \text{CL}(X) \) satisfy that

$$\theta(H(Tx, Ty)) \leq \varphi(d(x, y))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx,$$

(2.25)

where \( \theta \in \Theta \) and \( \varphi \) satisfies (2.7). Then for each \( x_0 \in X \), there exists an orbit \( \{x_n\}_{n \in \mathbb{N}} \) of \( T \) and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, \( z \in X \) is fixed point of \( T \) if and only if the function \( f \) defined by (1.8) is \( T \) orbitally lower semicontinuous at \( z \).

If \( \varphi(d(x, y)) \) in (2.6) is replaced by \( \varphi(d(x, Tx)) \), one has

**Theorem 2.5.** Let \( (X, d) \) be a complete metric space and \( T : X \to \text{CL}(X) \) satisfy that

$$\theta(d(y, Ty)) \leq \varphi(d(x, Tx))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx,$$

(2.26)

where \( \theta \in \Theta \) and \( \varphi \) satisfies (2.7). Then for each \( x_0 \in X \), there exists an orbit \( \{x_n\}_{n \in \mathbb{N}} \) of \( T \) and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, \( z \in X \) is fixed point of \( T \) if and only if the function \( f \) defined by (1.8) is \( T \) orbitally lower semicontinuous at \( z \).
Proof. Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.7), (2.26), and Lemma 2.2 that for $q = 1/\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}\}, 1/2 > 1$ there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq \frac{\theta(d(x_1, Tx_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}\}, 1/2} \leq \frac{\varphi(d(x_0, Tx_0))\theta(d(x_0, x_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}\}, 1/2} \leq \sqrt{\varphi(d(x_0, Tx_0))}\theta(d(x_0, x_1)), (2.27)$$

$$\theta(d(x_2,Tx_2)) \leq \varphi(d(x_1, Tx_1))\theta(d(x_1, x_2)) \leq \frac{\varphi(d(x_1, Tx_1))\theta(d(x_1, x_1))}{\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}\}, 1/2} \leq \sqrt{\varphi(d(x_1, Tx_1))}\theta(d(x_1, Tx_1)).$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n \in \mathbb{N}_0} \subset X$ satisfying $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$.

$$\theta(d(x_n, x_{n+1})) \leq \frac{\theta(d(x_n, Tx_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}\}, 1/(n + 1)} \leq \frac{\varphi(d(x_{n-1}, Tx_{n-1}))\theta(d(x_{n-1}, x_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}\}, 1/(n + 1)} \leq \sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}\theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, (2.28)$$

$$\theta(d(x_{n+1},Tx_{n+1})) \leq \varphi(d(x_n, Tx_n))\theta(d(x_n, x_{n+1})) \leq \frac{\varphi(d(x_n, Tx_n))\theta(d(x_n, Tx_n))}{\max\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}\}, 1/(n + 1)} \leq \sqrt{\varphi(d(x_n, Tx_n))}\theta(d(x_n, Tx_n)), \quad \forall n \in \mathbb{N}. (2.29)$$

Suppose that $x_{n_0} \in Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$. It is easy to verify that $x_n = x_{n_0}$ for all $n \geq n_0$ and the conclusion of Theorem 2.5 holds.
Suppose that \( x_n \notin T x_n \) for each \( n \in \mathbb{N}_0 \). It follows that \( \{d(x_n, T x_n)\}_{n \in \mathbb{N}_0} \) and \( \{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0} \) are positive sequences. Combining (2.7), (2.28), (2.29), (b) and (e), we infer that \( \{\theta(d(x_n, x_{n+1}))\}_{n \in \mathbb{N}_0} \) and \( \{\theta(d(x_n, T x_n))\}_{n \in \mathbb{N}_0} \) are both positive and decreasing, so do \( \{d(x_n, x_{n+1})\}_{n \in \mathbb{N}_0} \) and \( \{d(x_n, T x_n)\}_{n \in \mathbb{N}_0} \). It follows that there exist constants \( a, \beta, s \) and \( t \) satisfying

\[
\begin{align*}
\lim_{n \to \infty} \theta(d(x_n, x_{n+1})) &= \alpha \geq 0, \\
\lim_{n \to \infty} d(x_n, x_{n+1}) &= \beta \geq 0, \\
\lim_{n \to \infty} \theta(d(x_n, T x_n)) &= s \geq 0, \\
\lim_{n \to \infty} d(x_n, T x_n) &= t \geq 0.
\end{align*}
\] (2.30)

Notice that (2.7) implies that there exists a constant \( r \) such that

\[
\lim_{n \to \infty} \sup_{l \in [0, r]} \varphi(d(x_n, T x_n)) \leq \lim_{l \to r^+} \varphi(l) = r \in [0, 1).
\] (2.31)

Taking upper limits in (2.29) and by (2.30) and (2.31) we get that

\[
s \leq \sqrt{\lim_{n \to \infty} \sup_{l \in [0, r]} \varphi(d(x_n, T x_n)) \lim_{n \to \infty} \sup_{l \in [0, r]} \theta(d(x_n, T x_n))} \leq \sqrt{rs},
\] (2.32)

which implies that \( s = 0 \), which together with (2.30) and (a) ensures that

\[
0 \leq \theta(t) < \theta(d(x_n, T x_n)) \longrightarrow 0, \quad n \longrightarrow \infty,
\] (2.33)

that is, \( \theta(t) = 0 \), which gives that \( t = 0 \) by (b) and (d). It follows from (2.28), (2.30), and (2.31) that

\[
\alpha \leq \sqrt{\lim_{n \to \infty} \sup_{l \in [0, r]} \varphi(d(x_n, T x_n)) \lim_{n \to \infty} \sup_{l \in [0, r]} \theta(d(x_{n-1}, x_n))} \leq \sqrt{r}\alpha,
\] (2.34)

which yields that \( \alpha = 0 \). Notice that (2.30) and (a) guarantee that

\[
0 \leq \theta(\beta) < \theta(d(x_n, x_{n+1})) \longrightarrow 0, \quad n \longrightarrow \infty,
\] (2.35)

which together with (b) and (d) yields that \( \beta = 0 \). The result of the proof is similar to that of Theorem 2.3 and is omitted. This completes the proof. \( \square \)

The result below follows from Theorem 2.5.

**Theorem 2.6.** Let \( (X,d) \) be a complete metric space and \( T : X \to \text{CL}(X) \) satisfy that

\[
\theta(H(Tx,Ty)) \leq \varphi(d(x,Tx))\theta(d(x,y)), \quad \forall (x,y) \in X \times Tx,
\] (2.36)

where \( \theta \in \Theta \) and \( \varphi \) satisfies (2.7). Then for each \( x_0 \in X \), there exists an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \) of \( T \) and \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). Furthermore, \( z \in X \) is fixed point of \( T \) if and only if the function \( f \) defined by (1.8) is \( T \) orbitally lower semicontinuous at \( z \).
3. Comparisons and Examples

Now we construct two examples to compare the results in Section 2 with the corresponding results in [6–8].

Remark 3.1. Theorems 2.3 and 2.4 extend Theorems 1.1–1.3, and Theorems 2.5 and 2.6 are different from Theorems 1.1–1.3, respectively, in the following ways:

1. the ranges $CL(X)$ of the nonlinear set-valued contractive mappings $T$ in Theorems 2.3–2.6 are more general than the ranges $C(X)$ and $CB(X)$ of the set-valued contraction mappings $T$ in Theorems 1.1–1.3, respectively;

2. the $T$ orbit lower semicontinuity at some $z \in X$ of the functions $f(x) = d(x, Tx)$ in Theorems 2.3 and 2.4 is weaker than the continuity of the set-valued contraction mappings $T$ in $X$ in Theorems 1.1–1.3, respectively;

3. the set-valued contraction mappings (1.1) and (1.2) are special cases of the nonlinear set-valued contractive mapping (2.6) with $\theta \equiv 1$ because

$$d(y, Ty) \leq H(Tx, Ty), \quad \forall (x, y) \in X \times Tx. \quad (3.1)$$

Example 3.2 below shows that Theorems 2.3 and 2.4 extend substantively Theorems 1.1–1.3, respectively.

Example 3.2. Let $X = (-\infty, 3/10]$ and $d$ be the standard metric in $X$. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to [0, 1)$ and $T : X \to CL(X)$ be defined by

$$\theta(t) = t^{1/2}, \quad \varphi(t) = \frac{2\sqrt{6}}{5}, \quad \forall t \in \mathbb{R}^+, \quad Tx = \begin{cases} (-\infty, \frac{1}{4}x), & \forall x \in (-\infty, 0), \\ [0, 2x^2], & \forall x \in [0, \frac{3}{10}], \end{cases} \quad (3.2)$$

respectively. It is clear that $\theta \in \Theta$, $\varphi$ satisfies (2.7) and

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (-\infty, 0) \\ x - 2x^2, & \forall x \in [0, \frac{3}{10}] \end{cases} \quad (3.3)$$

is $T$ orbitally lower semicontinuous in $X$. In order to prove (2.6) holds, we consider two possible cases.

Case 1. Let $x \in (-\infty, 0)$ and $y \in Tx = (-\infty, (1/4)x]$. It is clear that

$$\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) = \frac{1}{2}\theta(d(x, y)) \leq \varphi(d(x, y))\theta(d(x, y)). \quad (3.4)$$
Case 2. Let $x \in [0, 3/10]$ and $y \in Tx = [0, 2x^2]$. It follows that
\[
\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) = \sqrt{2}|x + y|^{1/2}\theta(d(x, y))
\leq \sqrt{2}\left(\frac{3}{10} + \frac{9}{50}\right)^{1/2}\theta(d(x, y)) = \varphi(d(x, y))\theta(d(x, y)),
\]
that is, (2.6) holds. Therefore all assumptions of Theorems 2.3 and 2.4 are satisfied. It follows from each of Theorems 2.3 and 2.4 that $T$ has a fixed point in $X$. However, we cannot invoke any one of Theorems 1.1–1.3 to show the existence of fixed points for the mapping $T$ in $X$. Indeed, taking $x_0 = 3/10$ and $y_0 = 1/5$, we get that
\[
H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \leq \frac{r}{10} = rd(x_0, y_0),
\]
for any $r \in (0, 1)$ and
\[
H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \leq \frac{1}{10}\varphi\left(\frac{1}{10}\right) = \varphi(d(x_0, y_0))d(x_0, y_0),
\]
for any mapping $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$ with each of (1.3) and (1.4).

Next we construct an example to explain Theorems 2.5 and 2.6.

Example 3.3. Let $X = [-3/10, +\infty)$ and $d$ be the standard metric in $X$. Define $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \rightarrow [0, 1)$ and $T : X \rightarrow CL(X)$ by
\[
\theta(t) = t^{1/2}, \quad \forall t \in \mathbb{R}^+, \quad \varphi(t) = \begin{cases} 2\sqrt{2}t^{1/2}, & \forall t \in \left(0, \frac{1}{8}\right), \\ 2\sqrt{6}, & \forall t \in \{0\} \cup \left[\frac{1}{8}, +\infty\right), \end{cases}
\]
\[
Tx = \begin{cases} \left[\frac{x}{4(1 + x)}\right], & \forall x \in (0, +\infty), \\ [-2x^2, 0], & \forall x \in \left[-\frac{3}{10}, 0\right], \end{cases}
\]
respectively. It is easy to see that (2.7) holds and
\[
f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (0, +\infty), \\ -2x^2 - x, & \forall x \in \left[-\frac{3}{10}, 0\right], \end{cases}
\]
is $T$ orbitally lower semicontinuous in $X$. In order to check (2.26), we have to consider two cases as follows.
Case 1. Let $x \in (0, +\infty)$ and $y \in Tx = \{x/4(1 + x), +\infty\}$. It is clear that

$$
\theta(d(y, Ty)) = 0 \leq \theta(H(Tx, Ty)) = \left| \frac{x}{4(1 + x)} - \frac{y}{4(1 + y)} \right|^{1/2}
$$

$$
= \frac{\theta(d(x, y))}{2(1 + x)^{1/2}(1 + y)^{1/2}} \leq \frac{\theta(d(x, y))}{2(1 + x)^{1/2}(1 + x/4(1 + x))^{1/2}}
$$

$$
= \frac{\theta(d(x, y))}{(5x + 4)^{1/2}} \leq \frac{\theta(d(x, y))}{2} \leq \frac{2\sqrt{6}}{5}\theta(d(x, y))
$$

$$
= \varphi(0)\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).
$$

Case 2. Let $x \in [-3/10, 0]$ and $y \in Tx = [-2x^2, 0]$. It follows that

$$
\theta(d(y, Ty)) \leq \theta(H(Tx, Ty)) = \sqrt{2}|x + y|^{1/2}\theta(d(x, y)) \leq \sqrt{2}|x - 2x^2|^{1/2}\theta(d(x, y)).
$$

(3.11)

For $x = 0$, we have

$$
\sqrt{2}\left| x - 2x^2 \right|^{1/2}\theta(d(x, y)) = 0 \leq \varphi(d(x, Tx))\theta(d(x, y)).
$$

(3.12)

For $x \in [-3/10, -1/4) \cup (-1/4, 0)$, we infer that

$$
\sqrt{2}\left| x - 2x^2 \right|^{1/2}\theta(d(x, y)) \leq 2\sqrt{2}\left(-2x^2 - x\right)^{1/2}\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).
$$

(3.13)

For $x = -1/4$, we get that

$$
\sqrt{2}\left| x - 2x^2 \right|^{1/2}\theta(d(x, y)) = \frac{\sqrt{3}}{2}\theta(d(x, y)) \leq \varphi\left(\frac{1}{5}\right)\theta(d(x, y)) = \varphi(d(x, Tx))\theta(d(x, y)).
$$

(3.14)

Hence (2.26) holds. Thus all assumptions of Theorems 2.5 and 2.6 are satisfied. It follows from each of Theorems 2.5 and 2.6 that $T$ has a fixed point in $X$.

Taking $x_0 = 1$ and $y_0 = -3/10$, we deduce that

$$
H(Tx_0, Ty_0) = H\left(\left[\frac{1}{8}, +\infty\right), \left[-\frac{9}{50}, 0\right]\right) = +\infty \leq \frac{13r}{10} = rd(x_0, y_0),
$$

(3.15)

for any $r \in (0, 1)$, and

$$
H(Tx_0, Ty_0) = +\infty \leq \frac{2\sqrt{6}}{5} \cdot \frac{13}{10} = \varphi(d(x_0, y_0))d(x_0, y_0),
$$

(3.16)
for any mapping $\varphi : \mathbb{R}^+ \to [0, 1]$ with each of (1.3) and (1.4). That is, Theorems 1.1–1.3 are inapplicable in proving the existence of fixed points for the nonlinear set-valued contractive mapping $T$.

References

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