Research Article

Best Periodic Proximity Points for Cyclic Weaker Meir-Keeler Contractions

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The purpose of this paper is to present the existence of the best period proximity point for cyclic weaker Meir-Keeler contractions and asymptotic cyclic weaker Meir-Keeler contractions in metric spaces.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$ we denote the set of all nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers. Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. Consider a mapping $f : A \cup B \to A \cup B$. $f$ is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. A point $x$ in $A$ is called a best proximity point of $f$ in $A$ if $d(x,f(x)) = d(A,B)$ is satisfied, where $d(A,B) = \inf \{d(x,y) : x \in A, y \in B \}$, and $x \in A$ is called a best periodic proximity point of $f$ in $A$ if $d(x,f^{2^k+1}x) = d(A,B)$ is satisfied, for some $k \in \mathbb{N} \cup \{0\}$. In 2005, Eldred et al. [1] proved the existence of a best proximity point for relatively nonexpansive mappings using the notion of proximal normal structure. In 2006, Eldred and Veeramani [2] proved the following existence theorem.

**Theorem 1.1** (see Theorem 3.10 in [2]). Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $f : A \cup B \to A \cup B$ is a cyclic contraction, that is, $f(A) \subseteq B$ and $f(B) \subseteq A$, and there exists $k \in (0,1)$ such that

$$d(fx,fy) \leq kd(x,y) + (1-k)d(A,B) \quad \text{for every} \ x \in A, y \in B. \quad \quad \quad (1.1)$$

Then there exists a unique best proximity point in $A$. Further, for each $x \in A$, $\{f^{2^k}x\}$ converges to the best proximity point.
In this paper, we also recall the notion of Meir-Keeler type mapping. A mapping $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler-type mapping (see [3]) if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$.

In the recent, Eldred et al. [1] introduced the below notion of cyclic Meir-Keeler contraction.

**Definition 1.2** (see [1]). Let $(X, d)$ be a metric space, and let $A$ and $B$ be nonempty subsets of $X$. Then $f: A \cup B \to A \cup B$ is called a cyclic Meir-Keeler contraction if the following are satisfied:

(i) $f(A) \subset B$ and $f(B) \subset A$;

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < d(A, B) + \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) < d(A, B) + \varepsilon$$

for all $x \in A$ and $y \in B$.

In the recent, Di Bari et al. [4] proved the following best proximity point theorem.

**Theorem 1.3** (see [4]). Let $X$ be a uniformly convex Banach space, and let $A$ and $B$ be nonempty subsets of $X$. Suppose $A$ is closed and convex and $f: A \cup B \to A \cup B$ is a cyclic Meir-Keeler contraction. Then there exists a unique best proximity point in $A$. Further, for each $x \in A$, \{f^{2n}x\} converges to best proximity point.

Later, many authors studied this subject, and many results on best proximity points are proved. (see, e.g., [5–10]). In this study, we will introduce the new concepts of cyclic weaker Meir-Keeler contractions and asymptotic cyclic weaker Meir-Keeler contractions in metric spaces, and the purpose of this paper is to present the existence of the best periodic proximity point for these contractions.

### 2. The Best Periodic Proximity Points for Cyclic Weaker Meir-Keeler Contractions

In this section, we first introduce the below notions of the weaker Meir-Keeler-type mapping, $q$-mapping, and cyclic weaker Meir-Keeler contraction in metric spaces.

**Definition 2.1**. Let $(X, d)$ be a metric space, and $q: \mathbb{R}^+ \to \mathbb{R}^+$. Then $q$ is called a weaker Meir-Keeler-type mapping in $X$ if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \leq d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $q^{n_0}(d(x, y)) < \eta$.

The following provides an example of a weaker Meir-Keeler-type mapping that is not a Meir-Keeler-type mapping in a metric space $(X, d)$.

**Example 2.2**. Let $X = \mathbb{R}^2$, and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1, x_2), \ y = (y_1, y_2) \in X.$$
If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \),
\[
\varphi(t) = \begin{cases} 
0 & \text{if } t \leq 1, \\
2t & \text{if } 1 < t < 2, \\
1 & \text{if } t \geq 2,
\end{cases}
\]
where \( t = d(x, y) \), \( x, y \in X \), then \( \varphi \) is a weaker Meir-Keeler-type mapping that is not a Meir-Keeler-type mapping in \( X \).

**Definition 2.3.** Let \((X, d)\) be a metric space. A mapping \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is called a \( \varphi \)-mapping in \( X \) if the mapping \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfies the following conditions:

1. \( \varphi \) is a weaker Meir-Keeler-type mapping in \( X \);
2. for all \( t > 0 \), \( \{\varphi^n(t)\}_{n=1}^{\infty} \) is nonincreasing;
3. for all \( t > 0 \), \( \varphi(t) > 0 \) and \( \varphi(0) = 0 \).

The following provides two examples of a \( \varphi \)-mapping.

**Example 2.4.** Let \( X = \mathbb{R}^2 \), and we define \( d : X \times X \to \mathbb{R}^+ \) by
\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1, x_2), \ y = (y_1, y_2) \in X.
\]

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be
\[
\varphi(t) = \frac{1}{2} t \quad \forall t \in \mathbb{R}^+. 
\]

Then \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a \( \varphi \)-mapping in \( X \).

**Example 2.5.** Let \( X = [0,4] \), and we define \( d : X \times X \to \mathbb{R}^+ \) by
\[
d(x, y) = |x - y| \quad \forall x, y \in X.
\]

If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \),
\[
\varphi(t) = \begin{cases} 
\frac{3}{4} t & \text{if } 0 \leq t \leq 1, \\
2t & \text{if } 1 < t < 2, \\
1 & \text{if } 2 \leq t \leq 4,
\end{cases}
\]
where \( t = d(x, y) \), \( x, y \in X \), then \( \varphi \) is a \( \varphi \)-mapping in \( X \).

**Definition 2.6.** Let \((X, d)\) be a metric space, and let \( A \) and \( B \) be nonempty subsets of \( X \). Then \( f : A \cup B \to A \cup B \) is called a cyclic weaker Meir-Keeler contraction if the following conditions hold:
(1) \( f(A) \subseteq B \) and \( f(B) \subseteq A \);

(2) there is a \( \varphi \)-mapping \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) in \( X \) such that for all \( n \in \mathbb{N} \) and \( x \in A, y \in B \) with \( d(x, y) - d(A, B) > 0 \),

\[
\begin{align*}
\varphi^n(d(x, y)) &< \varphi^n(d(x, y) - d(A, B)), \\
d(f^n x, f^n y) - d(A, B) &< \varphi^n(d(x, y) - d(A, B)), \\
d(x, y) - d(A, B) = 0 &\implies d(f^n x, f^n y) - d(A, B) = 0.
\end{align*}
\]  

(2.7)

The following provides an example of a cyclic weaker Meir-Keeler contraction.

**Example 2.7.** Let \( A = [-2,0] \) and \( B = [0,2] \) in the metric space \( (\mathbb{R}, d) \), where \( d(x, y) = |x - y| \). Define

\[
f(x) = -\frac{x}{4} \quad \forall x \in A \cup B.
\]  

(2.8)

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be defined by

\[
\varphi(t) = \begin{cases} 
3t/4 & \text{if } 0 \leq t \leq 1, \\
2t & \text{if } 1 < t < 2, \\
1 & \text{if } 2 \leq t \leq 4,
\end{cases}
\]  

(2.9)

where \( t = d(x, y), x \in A, y \in B \). Then all conditions (1) and (2) of Definition 2.6 and therefore \( f \) are a cyclic weaker Meir-Keeler contraction. Notice that \( d(A, B) = 0 \).

Now, we are in this position to state the following results.

**Lemma 2.8.** Let \( (X, d) \) be a metric space, and let \( A, B \) be nonempty subsets of \( X \). Suppose \( f : A \cup B \to A \cup B \) is a cyclic weaker Meir-Keeler contraction. Then \( \lim_{n \to \infty} d(f^n x, f^{n+1} x) = d(A, B) \) holds.

**Proof.** Since \( f : A \cup B \to A \cup B \) is a cyclic weaker Meir-Keeler contraction, there is a \( \varphi \)-mapping \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) in \( X \) such that

\[
d(f^n x, f^n y) - d(A, B) < \varphi^n(d(x, y) - d(A, B)),
\]  

(2.10)

for all \( n \in \mathbb{N} \) and \( x \in A, y \in B \).

Since \( \{\varphi^n(d(x, y))\}_{n \in \mathbb{N}} \) is nonincreasing, hence we also conclude \( \{\varphi^n(d(x, y) - d(A, B))\}_{n \in \mathbb{N}} \) is nonincreasing, and it must converge to some \( \eta \geq 0 \). We claim that \( \eta = 0 \). On the contrary, assume that \( \eta > 0 \). By the definition of the weaker Meir-Keeler-type mapping \( \varphi \), corresponding to \( \eta \) use, there exists \( \delta > 0 \) such that for \( x, y \in X \) with \( \eta \leq d(x, y) - d(A, B) < \delta + \eta \), there exists \( n_0 \in \mathbb{N} \) such that \( \varphi^{n_0}(d(x, y) - d(A, B)) < \eta \). Since \( \lim_{n \to -\infty} \varphi^n(d(x, y) - d(A, B)) = \eta \), there exists \( m_0 \in \mathbb{N} \) such that \( \eta \leq \varphi^{m_0}(d(x, y) - d(A, B)) < \delta + \eta \), for all \( m \geq m_0 \). Thus, we conclude that \( \varphi^{m_0+n_0}(d(x, y) - d(A, B)) < \eta \). So we get a contradiction. So \( \lim_{n \to -\infty} \varphi^n(d(x, y) - d(A, B)) = 0 \), and so \( \lim_{n \to -\infty} d(f^n x, f^n y) - d(A, B) = 0 \), that
Example 3.2. If \( f \) is a \( \text{asymptotic cyclic weaker Meir-Keeler contraction} \) in a metric space \((X,d)\), then \( d(f^n x, f^n y) = d(A, B) \). Thus, we also conclude that \( \lim_{n \to \infty} d(f^n x, f^{n+1} x) = d(A, B) \).

Applying above Lemma 2.8, it is easy to conclude the following theorem.

**Theorem 2.9.** Let \((X,d)\) be a metric space, and let \( A, B \) be nonempty subsets of \( X \). Suppose \( f : A \cup B \to A \cup B \) is a cyclic weaker Meir-Keeler contraction and if for some \( x \in A \), the sequence \( \{f^{2n+1} x\} \) converges to \( \bar{x} \in A \), then \( \bar{x} \) is a best periodic proximity point of \( f \) in \( A \).

**Proof.** By the definition of the weaker Meir-Keeler-type mapping \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) in \( X \), there exists \( n_0 \in \mathbb{N} \) such that \( \varphi^{n_0} (\eta) \leq \eta \) for each \( \eta > 0 \). Since \( \{f^{2n+1} x\} \) converges to \( \bar{x} \in A \), corresponding to above \( n_0 \) use, we have

\[
d(A, B) \leq d(\bar{x}, f^{2n_0+1} \bar{x})
\]

\[
\leq d(\bar{x}, f^{2n_0+1} x) + d(f^{2n_0+1} x, f^{2n_0+1} \bar{x}) - d(A, B) + d(A, B)
\]

\[
\leq d(\bar{x}, f^{2n_0+1} x) + \varphi^{2n_0+1} (d(f^{2(n-n_0)} x, \bar{x}) - d(A, B)) + d(A, B)
\]

\[
\leq d(\bar{x}, f^{2n_0+1} x) + \varphi^{2n_0} (d(f^{2(n-n_0)} x, \bar{x}) - d(A, B)) + d(A, B)
\]

\[
\leq d(\bar{x}, f^{2n_0+1} x) + d(f^{2(n-n_0)} x, \bar{x}) - d(A, B) + d(A, B)
\]

\[
\leq d(\bar{x}, f^{2n_0+1} x) + d(f^{2(n-n_0)} x, f^{2(n-n_0)+1} x) + d(f^{2(n-n_0)+1} x, \bar{x}).
\]

Letting \( n \to \infty \). Then \( d(A, B) = d(\bar{x}, f^{2n_0+1} \bar{x}) \). Thus \( \bar{x} \) is a best period proximity point of \( f \) in \( A \).

\[\square\]

### 3. The Best Periodic Proximity Points for Asymptotic Cyclic Weaker Meir-Keeler Contractions

In this section, we introduce the below notions of the asymptotic cyclic weaker Meir-Keeler-type sequence and asymptotic cyclic weaker Meir-Keeler contraction in a metric space \((X,d)\).

**Definition 3.1.** Let \((X,d)\) be a metric space. A sequence \( \{\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \}_{n \in \mathbb{N}} \) in \( X \) is called an asymptotic weaker Meir-Keeler-type sequence if \( \{\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \}_{n \in \mathbb{N}} \) satisfies the following conditions:

- \((C_1)\) for each \( \eta > 0 \), there exists \( \delta > 0 \) such that for \( x, y \in X \) with \( \eta \leq d(x, y) \leq \delta + \eta \), there exists \( 2n_0 \in \mathbb{N} \) such that \( \varphi_{2n_0} (d(x, y)) < \eta \);
- \((C_2)\) for all \( n \in \mathbb{N} \) and \( t > 0 \), \( \{\varphi_n(t)\}_{n \in \mathbb{N}} \) is nonincreasing;
- \((C_3)\) for all \( n \in \mathbb{N} \), \( \varphi_n(0) = 0 \) and \( \varphi_n(t) > 0, t > 0 \).

**Example 3.2.** Let \( X = \mathbb{R}^2 \) and we define \( d : X \times X \to \mathbb{R}^+ \) by

\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1, x_2), \ y = (y_1, y_2) \in X.
\]
Let \( \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \) be

\[
\varphi_n(t) = \frac{1}{2^n} t \quad \forall t \in \mathbb{R}^+, \, n \in \mathbb{N},
\]

where \( t = d(x, y), \, x, y \in X \). Then \( \{ \varphi_n | \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \}_{n \in \mathbb{N}} \) is an asymptotic weaker Meir-Keeler-type sequence in a metric space \((X, d)\).

**Definition 3.3.** Let \((X, d)\) be a metric space, and let \(A\) and \(B\) be nonempty subsets of \(X\). Then \(f : A \cup B \to A \cup B\) is an asymptotic cyclic weaker Meir-Keeler contraction if the following conditions hold:

1. \( f(A) \subset B \) and \( f(B) \subset A \);
2. there is an asymptotic weaker Meir-Keeler-type sequence \( \{ \varphi_n | \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \}_{n \in \mathbb{N}} \)
   such that for all \( n \in \mathbb{N} \) and \( x \in A, \, y \in B \) with \( d(x, y) - d(A, B) > 0 \),

   \[
d(f^n x, f^n y) - d(A, B) < \varphi_n(d(x, y) - d(A, B)),
   \]

   \[
d(x, y) - d(A, B) = 0 \quad \text{implies} \quad d(f^n x, f^n y) - d(A, B) = 0.
\]

Now, we are in this position to state the following results.

**Lemma 3.4.** Let \((X, d)\) be a metric space and \(A, B\) nonempty subsets of \(X\). Suppose \(f : A \cup B \to A \cup B\) is an asymptotic cyclic weaker Meir-Keeler contraction. Then \(\lim_{n \to \infty} d(f^n x, f^{n+1} x) = d(A, B)\) holds.

**Proof.** Since \(f : A \cup B \to A \cup B\) is an asymptotic cyclic weaker Meir-Keeler contraction, there is an asymptotic weaker Meir-Keeler-type sequence \( \{ \varphi_n | \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+ \}_{n \in \mathbb{N}} \) such that

\[
d(f^n x, f^n y) - d(A, B) < \varphi_n(d(x, y) - d(A, B)),
\]

for all \( n \in \mathbb{N} \) and \( x \in A, \, y \in B \).

Since \( \{ \varphi_n(d(x, y)) \}_{n \in \mathbb{N}} \) is nonincreasing, hence we also conclude \( \{ \varphi_n(d(x, y) - d(A, B)) \}_{n \in \mathbb{N}} \) is nonincreasing, and it must converge to some \( \eta \geq 0 \). We claim that \( \eta = 0 \). On the contrary, assume that \( \eta > 0 \). By the definition of asymptotic weaker Meir-Keeler-type sequence, corresponding to \( \eta \) use, there exists \( \delta > 0 \) such that for \( x, y \in X \) with \( \eta \leq d(x, y) - d(A, B) < \delta + \eta \), there exists \( 2n_0 \in \mathbb{N} \) such that \( \varphi_{2n_0}(d(x, y) - d(A, B)) < \eta \). Since \( \lim_{n \to \infty} \varphi_n(d(x, y) - d(A, B)) = \eta \), there exists \( m_0 \in \mathbb{N} \) such that \( \eta \leq \varphi_m(d(x, y) - d(A, B)) < \delta + \eta \), for all \( m \geq m_0 \). Thus, we conclude that \( \varphi_{m+2n_0}(d(x, y) - d(A, B)) < \eta \). So we get a contradiction. Therefore, \( \lim_{n \to \infty} d(f^n x, f^n y) - d(A, B) = 0 \), and so \( \lim_{n \to \infty} d(f^n x, f^n y) - d(A, B) = 0 \), that is, \( \lim_{n \to \infty} d(f^n x, f^n y) = d(A, B) \). Thus, we also conclude that \( \lim_{n \to \infty} d(f^n x, f^{n+1} x) = d(A, B) \).

Applying above Lemma 3.4, we are easy to conclude the following theorem.

**Theorem 3.5.** Let \((X, d)\) be a metric space and \(A, B\) nonempty subsets of \(X\). Suppose \(f : A \cup B \to A \cup B\) is an asymptotic cyclic weaker Meir-Keeler contraction, and if for some \( x \in A \), the sequence \( \{ f^{2n+1} x \} \) converges to \( \overline{x} \in A \), then \( \overline{x} \) is a best periodic proximity point of \( f \) in \( A \).
Proof. By the definition of the asymptotic weaker Meir-Keeler-type sequence \( \{ \varphi_n \mid \varphi_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \} \) as \( \eta \) increases, there exists \( 2n_0 \in \mathbb{N} \) such that \( \varphi_{2n_0}(\eta) \leq \eta \) for each \( \eta > 0 \). Since \( \{ f^{2n+1}x \} \) converges to \( \bar{x} \in A \), corresponding to above \( 2n_0 \) use, we have

\[
d(A, B) \leq d(\bar{x}, f^{2n_0+1}\bar{x}) \\
\leq d(\bar{x}, f^{2n+1}x) + d(f^{2n+1}x, f^{2n_0+1}\bar{x}) - d(A, B) + d(A, B) \\
\leq d(\bar{x}, f^{2n+1}x) + \varphi_{2n_0}(d(f^{2(n-n_0)}x, \bar{x}) - d(A, B) + d(A, B) \\
\leq d(\bar{x}, f^{2n+1}x) + d(f^{2(n-n_0)}x, \bar{x}) - d(A, B) + d(A, B) \\
\leq d(\bar{x}, f^{2n+1}x) + d(f^{2(n-n_0)}x, f^{2(n-n_0)+1}x) + d(f^{2(n-n_0)+1}x, \bar{x}).
\]

Letting \( n \to \infty \). Then \( d(A, B) = d(\bar{x}, f^{2n_0+1}\bar{x}) \). Thus \( \bar{x} \) is a best period proximity point of \( f \) in \( A \). \( \square \)

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