Research Article

Strong Convergence Theorems for the Generalized Split Common Fixed Point Problem

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We introduce the generalized split common fixed point problem (GSCFPP) and show that the GSCFPP for nonexpansive operators is equivalent to the common fixed point problem. Moreover, we introduce a new iterative algorithm for finding a solution of the GSCFPP and obtain some strong convergence theorems under suitable assumptions.

1. Introduction

Let $H_1$ and $H_2$ be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given integers $p, r \geq 1$, let us recall that the multiple-set split feasibility problem (MSSFP) was recently introduced [1] and is to find a point:

$$x^* \in \bigcap_{i=1}^{p} C_i, \quad Ax^* \in \bigcap_{j=1}^{r} Q_j,$$  \hspace{1cm} (1.1)

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of $H_1$ and $H_2$, respectively. If $p = r = 1$, the MSSFP (1.1) becomes the so-called split feasibility problem (SFP) [2] which is to find a point:

$$x^* \in C, \quad Ax^* \in Q,$$  \hspace{1cm} (1.2)

where $C$ and $Q$ are nonempty closed convex subsets of $H_1$ and $H_2$, respectively. Recently, the SFP (1.2) and MSSFP (1.1) have been investigated by many researchers; see, [3–10].
Since every closed convex subset in a Hilbert space is looked as the fixed point set of its associating projection, the MSSFP (1.1) becomes a special case of the split common fixed point problem (SCFPP), which is to find a point:

$$x^* \in \bigcap_{i=1}^{p} \text{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{r} \text{Fix}(T_j),$$

(1.3)

where \( U_i : H_1 \to H_1 \) (\( i = 1,2,\ldots,p \)) and \( T_j : H_2 \to H_2 \) (\( j = 1,2,\ldots,r \)) are nonlinear operators. If \( p = r = 1 \), the problem (1.3) reduces to the so-called two-set SCFPP, which is to find a point:

$$x^* \in \text{Fix}(U), \quad Ax^* \in \text{Fix}(T).$$

(1.4)

Censor and Segal in [11] firstly introduced the concept of SCFPP in finite-dimensional Hilbert spaces and considered the following iterative algorithm for the two-set SCFPP (1.4) for Class-\( \Gamma \) operators:

$$x_{n+1} = U(x_n - \gamma A^*(I - T)Ax_n), \quad n \geq 0,$$

(1.5)

where \( x_0 \in H_1, 0 < \gamma < 2/\|A\|^2 \) and \( I \) is the identity operator. They proved the convergence of the algorithm (1.5) to a solution of problem (1.4). Moreover, they introduced a parallel iterative algorithm, which converges to a solution of the SCFPP (1.3). However, the parallel iterative algorithm does not include the algorithm (1.5) as a special case.

Very recently, Wang and Xu in [12] considered the SCFPP (1.3) for Class-\( \Gamma \) operators and introduced the following iterative algorithm for solving the SCFPP (1.3):

$$x_{n+1} = U_{[s]}(x_n - \gamma A^*(I - T_{[n]})Ax_n), \quad n \geq 0.$$

(1.6)

Under some mild conditions, they proved some weak and strong convergence theorems. Their iterative algorithm (1.6) includes Censor and Segal’s algorithm (1.5) as a special case for the two-set SCFPP (1.4). Moreover, they prove that the SCFPP (1.3) for the Class-\( \Gamma \) operators is equivalent to a common fixed point problem. This is also a classical method. Many problems eventually converted to a common fixed point problem; see [13–15].

Motivated and inspired by the aforementioned research works, we introduce a generalized split common fixed point problem (GSCFPP) which is to find a point:

$$x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{\infty} \text{Fix}(T_j).$$

(1.7)

Then, we show that the GSCFPP (1.7) for nonexpansive operators is equivalent to the following common fixed point problem:

$$x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(U_i), \quad x^* \in \bigcap_{j=1}^{\infty} \text{Fix}(V_j),$$

(1.8)
where $V_j = I - \gamma A^*(I-T_j)A$ ($0 < \gamma \leq 1/\|A\|^2$) for every $j \in \mathbb{N}$. Moreover, we give a new iterative algorithm for solving the GSCFPP (1.7) for nonexpansive operators and obtain some strong convergence theorems.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to $x$ and converges strongly to $x$, respectively.

An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of $T$ is denoted by $F(T)$. It is known that $F(T)$ is closed and convex. An operator $f : H \to H$ is called contraction if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho \|x - y\|$ for all $x, y \in H$. Let $C$ be a nonempty closed convex subset of $H$. For each $x \in H$, there exists a unique nearest point in $C$, denoted by $P_Cx$, such that $\|x - P_Cx\| \leq \|x - y\|$ for every $y \in C$. $P_C$ is called a metric projection of $H$ onto $C$. It is known that for each $x \in H$,

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0 \quad (2.1)$$

for all $y \in C$.

Let $\{T_n\}$ be a sequence of operators of $H$ into itself. The set of common fixed points of $\{T_n\}$ is denoted by $F(\{T_n\})$, that is, $F(\{T_n\}) = \cap_{n=1}^{\infty} F(T_n)$. A sequence $\{T_n\}$ is said to be strongly nonexpansive if each $\{T_n\}$ is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \to 0 \quad (2.2)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $C$ such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$; see [16, 17]. A sequence $\{z_n\}$ in $H$ is said to be an approximate fixed point sequence of $\{T_n\}$ if $z_n - T_n z_n \to 0$. The set of all bounded approximate fixed point sequences of $\{T_n\}$ is denoted by $\bar{F}(\{T_n\})$; see [16, 17]. We know that if $\{T_n\}$ has a common fixed point, then $\bar{F}(\{T_n\})$ is nonempty; that is, every bounded sequence in the common fixed point set is an approximate fixed point sequence. A sequence $\{T_n\}$ with a common fixed point is said to satisfy the condition $(Z)$ if every weak cluster point of $\{x_n\}$ is a common fixed point whenever $\{x_n\} \in \bar{F}(\{T_n\})$. A sequence $\{T_n\}$ of nonexpansive mappings of $H$ into itself is said to satisfy the condition $(R)$ if

$$\lim_{n \to \infty} \sup_{y \in D} \|T_{n+1} y - T_n y\| = 0 \quad (2.3)$$

for every nonempty bounded subset $D$ of $H$; see [18].

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 (see [16]). Let $C$ be a nonempty subset of a Hilbert space $H$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of $C$ into $H$. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that $\lim\inf_{n \to \infty} \lambda_n > 0$. Let $\{U_n\}$ be a sequence of mappings of $C$ into $H$ defined by $U_n = \lambda_n I + (1 - \lambda_n)T_n$ for $n \in \mathbb{N}$, where $I$ is the identity mapping on $C$. Then $\{U_n\}$ is a strongly nonexpansive sequence.
Lemma 2.2 (see [16]). Let $H$ be a Hilbert space, $C$ a nonempty subset of $H$, and \{S_n\} and \{T_n\} sequences of nonexpansive self-mappings of $C$. Suppose that \{S_n\} or \{T_n\} is a strongly nonexpansive sequence and $\overline{F}(\{S_n\}) \cap \overline{F}(\{T_n\})$ is nonempty. Then $\overline{F}(\{S_n\}) \cap \overline{F}(\{T_n\}) = \overline{F}(\{S_nT_n\})$.

Lemma 2.3 (see [17]). Let $H$ be a Hilbert space, and $C$ a nonempty subset of $H$. Both \{S_n\} and \{T_n\} satisfy the condition (R) and \{T_ny : n \in \mathbb{N}, y \in D\} is bounded for any bounded subset $D$ of $C$. Then \{S_nT_n\} satisfies the condition (R).

Lemma 2.4 (see [19]). Let \{x_n\} and \{y_n\} be bounded sequences in a Banach space $X$ and let \{\beta_n\} be a sequence in $[0,1)$ with $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (2.4)$$

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.5 (see [20]). Assume that \{a_n\} is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad n \geq 0, \quad (2.5)$$

where \{\gamma_n\} is a sequence in $(0,1)$ and \{\delta_n\} is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

3. Main Results

Now we state and prove our main results of this paper.

Lemma 3.1. Let $A : H_1 \to H_2$ be a given bounded linear operator and let $T_n : H_2 \to H_2$ be a sequence of nonexpansive operators. Assume

$$A^{-1}(\text{Fix}(\{T_n\})) = \{x \in H_1 : Ax \in \text{Fix}(\{T_n\})\} \neq \emptyset. \quad (3.1)$$

For each constant $\gamma > 0$, $V_n$ is defined by the following:

$$V_n = I - \gamma A^*(I - T_n)A. \quad (3.2)$$

Then $\text{Fix}(\{V_n\}) = A^{-1}(\text{Fix}(\{T_n\}))$. Moreover, for $0 < \gamma \leq 1/\|A\|^2$, $V_n$ is nonexpansive on $H_1$ for $n \in \mathbb{N}$.
Proof. Since the inclusion $A^{-1}({\text{Fix}}([T_n])) \subseteq {\text{Fix}}([V_n])$ is evident, now we only need to show the converse inclusion. If $z \in {\text{Fix}}([V_n])$, then we have $A^*(I - T_n)Az = 0$. Since $A^{-1}({\text{Fix}}([T_n])) \neq \emptyset$, we take an arbitrary $p \in A^{-1}({\text{Fix}}([T_n]))$. Hence

$$
\|Az - T_nAz\|^2 = \langle Az - T_nAz, Az - T_nAz \rangle \\
= \langle Az - T_nAz, Az - Ap + Ap - T_nAz \rangle \\
= \langle A^*(I - T_n)Az, z - p \rangle + \langle Az - T_nAz, Ap - T_nAz \rangle \\
= -\frac{1}{2}\|Az - Ap\|^2 + \frac{1}{2}\|Az - T_nAz\|^2 + \frac{1}{2}\|Ap - T_nAz\|^2 \\
\leq \frac{1}{2}\|Az - T_nAz\|^2.
$$

(3.3)

It follows that $(1/2)\|Az - T_nAz\|^2 \leq 0$, then $Az = T_nAz$ for every $n \in \mathbb{N}$, hence $z \in A^{-1}({\text{Fix}}([T_n]))$. Next we turn to show that $V_n$ is a nonexpansive operator for $n \in \mathbb{N}$. Since $T_n$ is nonexpansive, we have

$$
\|(I - T_n)Ax - (I - T_n)Ay\|^2 = \|Ax - Ay\|^2 + \|T_nAx - T_nAy\|^2 - 2\langle Ax - Ay, T_nAx - T_nAy \rangle \\
\leq 2\|Ax - Ay\|^2 - 2\langle Ax - Ay, T_nAx - T_nAy \rangle \\
\leq 2\langle Ax - Ay, Ax - Ay - (T_nAx - T_nAy) \rangle.
$$

(3.4)

Hence

$$
\|V_nx - V_ny\|^2 = \|(I - \gamma A^*(I - T_n)A)x - (I - \gamma A^*(I - T_n)A)y\|^2 \\
= \|x - y\|^2 + \gamma^2\|A\|^2\|(I - T_n)Ax - (I - T_n)Ay\|^2 \\
- 2\gamma\langle Ax - Ay, (I - T_n)Ax - (I - T_n)Ay \rangle \\
\leq \|x - y\|^2 + \gamma\left(\gamma\|A\|^2 - 1\right)\|(I - T_n)Ax - (I - T_n)Ay\|^2.
$$

(3.5)

For $0 < \gamma \leq 1/\|A\|^2$, we can immediately obtain that $V_n$ is a nonexpansive operator for every $n \in \mathbb{N}$.

From Lemma 3.1, we can obtain that the solution set of GSCFPP (1.7) is identical to the solution set of problem (1.8).

**Theorem 3.2.** Let \{${U_n}$\} and \{${V_n}$\} be sequences of nonexpansive operators on Hilbert space $H_1$. Both \{${U_n}$\} and \{${V_n}$\} satisfy the conditions (R) and (Z). Let $f : H_1 \to H_1$ be a contraction with coefficient
\[ \rho \in [0,1). \text{ Suppose } \Omega = \text{Fix}(U_n) \cap \text{Fix}(V_n) \neq \emptyset. \text{ Take an initial guess } x_1 \in H_1 \text{ and define a sequence } \{x_n\} \text{ by the following algorithm:} \]

\[
y_n = \lambda_n x_n + (1 - \lambda_n) V_n x_n, \]

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n, \quad (3.6)\]

where \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, and \{\lambda_n\} are sequences in [0, 1]. If the following conditions are satisfied:

(i) \[ \alpha_n + \beta_n + \gamma_n = 1, \text{ for all } n \geq 1; \]

(ii) \[ \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \]

(iii) \[ 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1; \]

(iv) \[ 0 < \lim \inf_{n \to \infty} \lambda_n \leq \lim \sup_{n \to \infty} \lambda_n < 1; \]

(v) \[ \lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0, \]

then \{x_n\} converges strongly to \( w \in \Omega \) where \( w = P_\Omega f(w) \).

**Proof.** We proceed with the following steps.

**Step 1.** First show that there exists \( w \in \Omega \) such that \( w = P_\Omega f(w) \).

In fact, since \( f \) is a contraction with coefficient \( \rho \), we have

\[
\|P_\Omega f(x) - P_\Omega f(y)\| \leq \|f(x) - f(y)\| \leq \rho \|x - y\| \quad (3.7)
\]

for every \( x, y \). Hence \( P_\Omega f \) is also a contraction. Therefore, there exists a unique \( w \in \Omega \) such that \( w = P_\Omega f(w) \).

**Step 2.** Now we show that \( \{x_n\} \) is bounded.

Let \( p \in \Omega \), then \( p \in \text{Fix}(\{U_n\}) \) and \( p \in \text{Fix}(\{V_n\}) \). Hence

\[
\|U_n y_n - p\| \leq \|y_n - p\| \leq \lambda_n \|x_n - p\| + (1 - \lambda_n) \|V_n x_n - p\| \leq \|x_n - p\|. \quad (3.8)
\]

Then

\[
\|x_{n+1} - p\| \leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|U_n y_n - p\| \\
\leq \alpha_n \rho \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\
\leq (1 - \alpha_n (1 - \rho)) \|x_n - p\| + \alpha_n (1 - \rho) \frac{1}{1 - \rho} \|f(p) - p\| \quad (3.9)
\]

\[
\leq \max \left\{ \|x_n - p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\}.
\]

By induction on \( n \),

\[
\|V_n x_n - p\| \leq \|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\}. \quad (3.10)
\]
for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ and $\{V_n x_n\}$ are bounded, and hence, $\{U_n y_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded.

**Step 3.** We claim that $\tilde{F}([A_n]) = \tilde{F}([V_n])$ and $\tilde{F}([U_n A_n]) = \tilde{F}([U_n]) \cap \tilde{F}([V_n])$, where $A_n = \lambda_n I + (1 - \lambda_n)V_n$.

We first show the former equality. Let $\{z_n\}$ be a bounded sequence in $H$. If $\{z_n\} \in \tilde{F}([V_n])$, then

$$\|A_n z_n - z_n\| = \|\lambda_n z_n + (1 - \lambda_n)V_n z_n - z_n\| = (1 - \lambda_n)\|V_n z_n - z_n\| \to 0.$$  \hspace{1cm} (3.11)

Hence $\{z_n\} \in \tilde{F}([A_n])$. On the other hand, if $\{z_n\} \in \tilde{F}([A_n])$, combining (3.11) and $\limsup_{n \to \infty} \lambda_n < 1$, we obtain that $\|V_n z_n - z_n\| \to 0$. Hence $\{z_n\} \in \tilde{F}([V_n])$. Therefore, $\tilde{F}([A_n]) = \tilde{F}([V_n])$.

Next, we show the latter equality. Using Lemma 2.1, we know that $\{A_n\}$ is a strongly nonexpansive sequence. Thus, since $\tilde{F}([U_n]) \cap \tilde{F}([A_n]) = \tilde{F}([U_n]) \cap \tilde{F}([V_n]) \neq \emptyset$, from Lemma 2.2 we have

$$\tilde{F}([U_n A_n]) = \tilde{F}([U_n]) \cap \tilde{F}([A_n]) = \tilde{F}([U_n]) \cap \tilde{F}([V_n]).$$ \hspace{1cm} (3.12)

**Step 4.** $\{S_n\}$ satisfies the condition (R), where $S_n = U_n A_n$.

Let $D$ be a nonempty bounded subset of $H$. From the definition of $\{A_n\}$, we have, for all $y \in D$,

$$\|A_{n+1} y - A_n y\| = \|\lambda_{n+1} y + (1 - \lambda_{n+1})V_{n+1} y - \lambda_n y - (1 - \lambda_n)V_n y\|
\leq |\lambda_{n+1} - \lambda_n|\|y\| + \|V_{n+1} y - V_n y\| + |\lambda_{n+1} - \lambda_n|\|V_{n+1} y - \lambda_n V_n y\|
\leq |\lambda_{n+1} - \lambda_n|\|y\| + \|V_{n+1} y - V_n y\| + |\lambda_{n+1} - \lambda_n|\|V_{n+1} y - \lambda_n V_n y\|
+ |\lambda_n V_{n+1} y - \lambda_n V_n y|
= |\lambda_{n+1} - \lambda_n|\|y\| + \|V_{n+1} y - V_n y\| + |\lambda_{n+1} - \lambda_n|\|V_{n+1} y\|
+ \lambda_n\|V_{n+1} y - V_n y\|
= |\lambda_{n+1} - \lambda_n|\|y\| + \|V_{n+1} y\| + (1 + \lambda_n)\|V_{n+1} y - V_n y\|.$$  \hspace{1cm} (3.13)

It follows that

$$\sup_{y \in D}\|A_{n+1} y - A_n y\| \leq |\lambda_{n+1} - \lambda_n|\sup_{y \in D}(\|y\| + \|V_{n+1} y\|) + (1 + \lambda_n)\sup_{y \in D}\|V_{n+1} y - V_n y\|. \hspace{1cm} (3.14)$$

Since $\{V_n\}$ satisfies the condition (R) and $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$, we have

$$\lim_{n \to \infty} \sup_{y \in D}\|A_{n+1} y - A_n y\| = 0.$$  \hspace{1cm} (3.15)
Step 5. We show that, is, \( \{A_n\} \) satisfies the condition (\( R \)). Since \( \{A_n y : n \in \mathbb{N}, y \in D \} \) is bounded for any bounded subset \( D \) of \( H_1 \), by using Lemma 2.3, we have that \( \{V_n A_n\} \) satisfies the condition (\( R \)), that is, \( \{S_n\} \) satisfies the condition (\( R \)).

We can write (3.6) as \( x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n \) where \( z_n = (\alpha_n f(x_n) + \gamma_n \alpha_n S_n x_n) / 1 - \beta_n \). It follows that

\[
\begin{align*}
    z_{n+1} - z_n &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} S_{n+1} x_{n+1} - \frac{\alpha_n}{1 - \beta_n} f(x_n) - \frac{\gamma_n}{1 - \beta_n} S_n x_n \\
    &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - f(x_n)) + \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) \\
    &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} (S_{n+1} x_{n+1} - S_n x_n) + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) S_n x_n.
\end{align*}
\]  

From Step 2, we may assume that \( \{x_n\} \subset D' \), where \( D' \) is a bounded set of \( H_1 \). Then from (3.16), we obtain

\[
\begin{align*}
    \|z_{n+1} - z_n\| &\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|S_n x_n\|) + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \rho \|x_{n+1} - x_n\| \\
    &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} x_{n+1} - S_n x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|S_{n+1} x_{n+1} - S_n x_n\| \\
    &\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|S_n x_n\|) + \left[ 1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (1 - \rho) \right] \|x_{n+1} - x_n\| \\
    &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\|.
\end{align*}
\]  

It follows that

\[
\begin{align*}
    \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| (\|f(x_n)\| + \|S_n x_n\|) \\
    &\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\| - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (1 - \rho) \|x_{n+1} - x_n\|.
\end{align*}
\]  

Since \( \{S_n\} \) satisfies the condition (\( R \)), combining \( \alpha_n \to 0 \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \sup (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]  

Hence by Lemma 2.4, we get \( \|z_n - x_n\| \to 0 \) as \( n \to \infty \). Consequently,

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|z_n - x_n\| = 0.
\]
Step 6. We claim that \( \{x_n\} \in \tilde{F}([U_n]) \cap \tilde{F}([V_n]). \)

From (3.6), we have

\[
\|S_n x_n - x_n\| \leq \|S_n x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
= \|S_n x_n - \alpha_n f(x_n) - \beta_n x_n - \gamma_n S_n x_n\| + \|x_{n+1} - x_n\| \\
\leq \alpha_n \|S_n x_n - f(x_n)\| + \beta_n \|S_n x_n - x_n\| + \|x_{n+1} - x_n\|, 
\]

and hence

\[
(1 - \beta_n)\|S_n x_n - x_n\| \leq \alpha_n \|S_n x_n - f(x_n)\| + \|x_{n+1} - x_n\|. 
\]

Since \( \|x_{n+1} - x_n\| \to 0, \alpha_n \to 0 \) and \( \limsup_{n \to \infty} \beta_n < 1 \), we derive

\[
\|S_n x_n - x_n\| \to 0. 
\]

Thus (3.23) and Steps 2 and 3 imply that

\[
\{x_n\} \in \tilde{F}([S_n]) = \tilde{F}([U_n]) \cap \tilde{F}([V_n]). 
\]

Step 7. Show \( \limsup_{n \to \infty} \langle f(w) - w, x_n - w \rangle \leq 0 \), where \( w = P_{\Omega} f(w) \).

Since \( \{x_n\} \) is bounded, there exist a point \( v \in H_1 \) and a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that

\[
\limsup_{n \to \infty} \langle f(w) - w, x_n - w \rangle = \lim_{i \to \infty} \langle f(w) - w, x_{n_i} - w \rangle 
\]

and \( x_{n_i} \rightharpoonup v \). Since \( \{U_n\} \) and \( \{V_n\} \) satisfy the condition \( (Z) \), from Step 6, we have \( v \in F([U_n]) \cap F([V_n]) \). Using (2.1), we get

\[
\limsup_{n \to \infty} \langle f(w) - w, x_n - w \rangle = \lim_{i \to \infty} \langle f(w) - w, x_{n_i} - w \rangle \\
= \langle f(w) - w, v - w \rangle \leq 0. 
\]

Step 8. Show \( x_n \to w = P_{\Omega} f(w) \).
Since \(w \in \Omega\), using (3.8), we have

\[
\|x_{n+1} - w\|^2 = \langle \alpha_n (f(x_n) - w) + \beta_n (x_n - w) + \gamma_n (U_n y_n - w), x_{n+1} - w \rangle \\
\leq \alpha_n \langle f(x_n) - f(w), x_{n+1} - w \rangle + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
+ \beta_n \|x_n - w\| \cdot \|x_{n+1} - w\| + \gamma_n \|y_n - w\| \cdot \|x_{n+1} - w\| \\
\leq \frac{1}{2} \alpha_n \rho \left( \|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
+ \frac{1}{2} \beta_n \left( \|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \frac{1}{2} \gamma_n \left( \|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) \\
\leq \frac{1}{2} \left[ 1 - \alpha_n (1 - \rho) \right] \|x_n - w\|^2 + \frac{1}{2} \|x_{n+1} - w\|^2 + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle,
\]

which implies that

\[
\|x_{n+1} - w\|^2 \leq \left[ 1 - \alpha_n (1 - \rho) \right] \|x_n - w\|^2 + 2\alpha_n (1 - \rho) \frac{1}{1 - \rho} \|f(w) - w, x_{n+1} - w\|,
\]

for every \(n \in \mathbb{N}\). Consequently, according to Step 7, \(\rho \in [0, 1)\), and Lemma 2.5, we deduce that \(\{x_n\}\) converges strongly to \(w = P_\Omega(w)\). This completes the proof. \(\square\)

Combining Lemma 3.1 and Theorem 3.2, we can obtain the following strong convergence theorem for solving the GSCFPP (1.7).

**Theorem 3.3.** Let \(\{U_n\}\) and \(\{T_n\}\) be sequences of nonexpansive operators on Hilbert space \(H_1\) and \(H_2\), respectively. Both \(\{U_n\}\) and \(\{T_n\}\) satisfy the conditions (R) and (Z). Let \(f : H_1 \to H_1\) be a contraction with coefficient \(\rho \in [0, 1)\). Suppose that the solution set \(\Omega\) of GSCFPP (1.7) is nonempty. Take an initial guess \(x_1 \in H_1\) and define a sequence \(\{x_n\}\) by the following algorithm:

\[
y_n = x_n - \gamma (1 - \lambda_n) A^* (I - T_n) A x_n,
\]

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n,
\]

where \(\gamma \in (0, 1/\|A\|^2)\), and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}\) are sequences in \([0, 1]\). If the following conditions are satisfied:

(i) \(\alpha_n + \beta_n + \gamma_n = 1\), for all \(n \geq 1\);

(ii) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\);

(iii) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\);

(iv) \(0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < 1\);

(v) \(\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0\),

then \(\{x_n\}\) converges strongly to \(w \in \Omega\) where \(w = P_\Omega f(w)\).
Proof. Set \( V_n = I - \gamma A^*(I - T_n)A \). By Lemma 3.1, \( V_n \) is a nonexpansive operator for every \( n \in \mathbb{N} \). We can rewrite (3.29) as

\[
y_n = \lambda_n x_n + (1 - \lambda_n)V_n x_n,
\]

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n.
\]

We only need to prove that \( \{V_n\} \) satisfies the conditions (R) and (Z). Assume that \( D \) is a nonempty bounded subset of \( H \). For every \( y \in D \), we have

\[
\| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \| \leq \gamma \| A^*(I - T_{n+1})Ay - A^*(I - T_n)Ay \| \\
\leq \gamma \| A \| \| T_{n+1}(Ay) - T_n(Ay) \|.
\]

(3.31)

Since \( \{T_n\} \) satisfies the condition (R), and \( D' = \{Ay : y \in D\} \) is bounded, it follows from (3.31) that

\[
\sup_{y \in D'} \| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \| \leq \gamma \| A \| \sup_{y \in D} \| T_{n+1}(Ay) - T_n(Ay) \| \\
= \gamma \| A \| \sup_{z \in D'} \| T_{n+1}z - T_nz \| \to 0.
\]

(3.32)

Therefore, \( \{V_n\} \) satisfies the condition (R).

Assume that \( x_n \rightharpoonup z \) and \( x_n - V_n x_n \rightharpoonup 0 \); we next show that \( V_n z = z \). By using \( x_n - V_n x_n \rightharpoonup 0 \), we have \( A^*(I - T_n)Ax_n \rightharpoonup 0 \). Since \( A^{-1}(\text{Fix}(\{T_n\})) \neq \emptyset \), we choose an arbitrary point \( p \in A^{-1}(\text{Fix}(\{T_n\})) \); then for every \( n \in \mathbb{N} \),

\[
\| Ax_n - T_n Ax_n \|^2 = \langle Ax_n - T_n Ax_n, Ax_n - Ap + Ap - T_n Ax_n \rangle \\
= \langle A^*(I - T_n)Ax_n, x_n - p \rangle + \langle Ax_n - T_n Ax_n, Ap - T_n Ax_n \rangle \\
= \langle A^*(I - T_n)Ax_n, x_n - p \rangle - \frac{1}{2} \| Ax_n - Ap \|^2 + \frac{1}{2} \| Ax_n - T_n Ax_n \|^2 \\
+ \frac{1}{2} \| Ap - T_n Ax_n \|^2 \\
\leq \langle A^*(I - T_n)Ax_n, x_n - p \rangle + \frac{1}{2} \| Ax_n - T_n Ax_n \|^2.
\]

(3.33)

Hence

\[
\frac{1}{2} \| Ax_n - T_n Ax_n \|^2 \leq \langle A^*(I - T_n)Ax_n, x_n - p \rangle \to 0.
\]

(3.34)

Then we get \( Ax_n \in \tilde{F}(\{T_n\}) \). Since \( \{T_n\} \) satisfies the condition (Z) and \( Ax_n \rightharpoonup Az \), we have \( Az \in F(\{T_n\}) \). From Lemma 3.1, we have \( z \in \text{Fix}(\{V_n\}) \). \( \square \)
Let $T : H \to H$ be a nonexpansive mapping with a fixed point, and define $T_n = T$ for all $n \in \mathbb{N}$. Then $\{T_n\}$ satisfies the conditions (R) and (Z). Thus, one obtains the algorithm for solving the two-set SCFPP (1.4).

**Corollary 3.4.** Let $U$ and $T$ be nonexpansive operators on Hilbert space $H_1$ and $H_2$, respectively. Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0, 1)$. Suppose that the solution set $\Omega$ of SCFPP (1.4) is nonempty. Take an initial guess $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm in (3.29), where $\gamma \in (0, 1/\|A\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$. If the following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$;
(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;
(iv) $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < 1$;
(v) $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega f(w)$.

**Remark 3.5.** By adding more operators to the families $\{U_n\}$ and $\{T_n\}$ by setting $U_i = I$ for $i \geq p + 1$ and $T_j = I$ for $j \geq r + 1$, the SCFPP (1.3) can be viewed as a special case of the GSCFPP (1.7).

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**References**


