Research Article

Positive Solutions to a Generalized Second-Order Difference Equation with Summation Boundary Value Problem

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Received 1 December 2011; Accepted 22 February 2012

By using Krasnoselskii’s fixed point theorem, we study the existence of positive solutions to the three-point summation boundary value problem

\[ \Delta^2 u(t - 1) + a(t)f(u(t)) = 0, \quad t \in \{1, 2, \ldots, T\}, \]
\[ u(0) = \beta \sum_{s=1}^{\eta} u(s), \quad u(T+1) = \alpha \sum_{s=\eta}^{T} u(s), \]

where \( f \) is continuous, \( T \geq 3 \) is a fixed positive integer, \( \eta \in \{1, 2, \ldots, T - 1\}, 0 < \alpha < (2T + 2)/\eta(\eta + 1), 0 < \beta < (2T + 2 - a\eta(\eta + 1))/(2T - \eta + 1), \) and \( \Delta u(t - 1) = u(t) - u(t - 1) \). We show the existence of at least one positive solution if \( f \) is either superlinear or sublinear.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors; one may see the text books [3, 4] and the papers [5–10]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

\[ u(0) = 0, \quad u(T+1) = 0, \]
\[ u(0) = 0, \quad au(s) = u(T+1), \]
\[ u(0) = 0, \quad u(T+1) - au(s) = b, \]
\begin{equation}
\begin{aligned}
    u(0) - a\Delta u(0) = 0, & \quad u(T + 1) = \beta u(s), \\
    u(0) - a\Delta u(0) = 0, & \quad \Delta u(T + 1) = 0,
\end{aligned}
\end{equation}

and so forth.

In [5], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [6, 7]. In [8], X. Lin and W. Liu, using the properties of the associate Green’s function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

In [9], Zhang and Medina studied the existence of positive solutions for second-order boundary value problems of difference equations by applying Krasnoselskii’s fixed point theorem. In [10], Henderson and Thompson used lower and upper solution methods to study the existence of multiple solutions for second-order discrete boundary value problems.

We are interested in the existence of positive solutions of the following second-order difference equation with three-point summation boundary value problem (BVP):

\begin{equation}
\begin{aligned}
    \Delta^2 u(t - 1) + a(t)f(u(t)) = 0, & \quad t \in \{1, 2, \ldots, T\}, \\
    u(0) = \beta \sum_{s=1}^{\eta} u(s), & \quad u(T + 1) = \alpha \sum_{s=1}^{\eta} u(s),
\end{aligned}
\end{equation}

where \( f \) is continuous, \( T \geq 3 \) is a fixed positive integer, \( \eta \in \{1, 2, \ldots, T - 1\} \).

The aim of this paper is to give some results for existence of positive solutions to (1.2), assuming that \( 0 < \alpha < (2T + 2)/\eta(\eta + 1), 0 < \beta < (2T + 2 - a\eta(\eta + 1))/\eta(2T - \eta + 1) \), and \( f \) is either superlinear or sublinear. Set

\begin{equation}
    f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.
\end{equation}

Then \( f_0 = 0 \) and \( f_\infty = \infty \) correspond to the superlinear case, and \( f_0 = \infty \) and \( f_\infty = 0 \) correspond to the sublinear case. Let \( \mathbb{N} \) be the nonnegative integer; we let \( \mathbb{N}_{i,j} = \{ k \in \mathbb{N} | i \leq k \leq j \} \) and \( \mathbb{N}_p = \mathbb{N}_{0,p} \). By the positive solution of (1.2), we mean that a function \( u(t) : \mathbb{N}_{T+1} \to [0, \infty) \) and satisfies the problem (1.2).

Recently, Sithiwirattham [11] proved the existence of positive solutions for the boundary value problem with summation condition

\begin{equation}
\begin{aligned}
    \Delta^2 u(t - 1) + a(t)f(u(t)) = 0, & \quad t \in \{1, 2, \ldots, T\}, \\
    u(0) = 0, & \quad u(T + 1) = \alpha \sum_{s=1}^{\eta} u(s),
\end{aligned}
\end{equation}

where \( f \) is continuous, \( T \geq 3 \) is a fixed positive integer, \( \eta \in \{1, 2, \ldots, T - 1\} \), and \( 0 < \alpha < 2T + 2/\eta(\eta + 1) \).

Throughout this paper, we suppose the following conditions hold:

(A1) \( f \in C([0, \infty), [0, \infty)) \);

(A2) \( a \in C(\mathbb{N}_{T+1}, [0, \infty)) \) and there exists \( t_0 \in \mathbb{N}_{N,T+1} \) such that \( a(t_0) > 0 \).
The proof of the main theorem is based upon an application of the following Krasnoselskii’s fixed point theorem in a cone.

**Theorem 1.1** (see [12]). Let $E$ be a Banach space, and let $K \subset E$ be a cone. Assume $\Omega_1$, $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_2$, or

(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial \Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_2$.

Then $A$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

### 2. Preliminaries

We now state and prove several lemmas before stating our main results.

**Lemma 2.1.** Let $\beta \neq (2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1)$. Then, for $y \in C(\mathbb{N}_{T+1}, [0, \infty))$, the problem

$$\Delta^2 u(t) + y(t) = 0, \quad t \in N_{1,T},$$

$$u(0) = \beta \sum_{s=1}^\eta u(s), \quad u(T + 1) = \alpha \sum_{s=1}^\eta u(s),$$

has a unique solution

$$u(t) = \frac{\beta \eta(\eta + 1) + 2T(1 - \beta\eta)}{(2T + 2 - \alpha\eta(\eta + 1) - \beta\eta(2T - \eta + 1))} \sum_{s=1}^T (T - s + 1)y(s)$$

$$- \frac{\beta(T + 1) - (\beta - \alpha)t}{(2T + 2 - \alpha\eta(\eta + 1) - \beta\eta(2T - \eta + 1)) \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s - 1)y(s)}$$

$$- \sum_{s=1}^{t-1} (t - s)y(s), \quad t \in \mathbb{N}_{T+1}.$$  

**Proof.** From (2.1), we get

$$\Delta u(t) - \Delta u(t - 1) = -y(t),$$

$$\Delta u(t - 1) - \Delta u(t - 2) = -y(t - 1),$$

$$\vdots$$

$$\Delta u(1) - \Delta u(0) = -y(1).$$
We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), \quad t \in \mathbb{N}_T, \quad (2.5)$$

we denote $\sum_{s=p}^{q} y(s) = 0$, if $p > q$. Similarly, summing the above equation from $t = 0$ to $t = h$, we get

$$u(h + 1) = u(0) + (h + 1)\Delta u(0) - \sum_{s=1}^{h} (h + 1 - s)y(s), \quad h \in \mathbb{N}_T, \quad (2.6)$$

changing the variable from $h + 1$ to $t$, we have

$$u(t) = u(0) + t\Delta u(0) - \sum_{s=1}^{t-1} (t - s)y(s) : A + Bt - \sum_{s=1}^{t-1} (t - s)y(s), \quad t \in \mathbb{N}_{T+1}. \quad (2.7)$$

We sum (2.7) from $s = 1, 2, \ldots, \eta$,

$$\sum_{s=1}^{\eta} u(s) = \eta A + \frac{\eta(\eta + 1)}{2} B - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} (t - s)y(s)$$

$$= \eta A + \frac{\eta(\eta + 1)}{2} B - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)y(s). \quad (2.8)$$

By (2.2) from $u(0) = \beta \sum_{s=1}^{\eta} u(s)$, we get

$$(1 - \beta \eta) A - \frac{\beta \eta(\eta + 1)}{2} B = -\frac{\beta^{\eta-1}}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)y(s), \quad (2.9)$$

and from $u(T + 1) = \alpha \sum_{s=1}^{\eta} u(s)$, we obtain

$$(1 - \alpha \eta) A + \left( T + 1 - \frac{\alpha \eta(\eta + 1)}{2} \right) B = \sum_{s=1}^{T} (T - s + 1)y(s) - \frac{\alpha^{\eta-1}}{2} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)y(s). \quad (2.10)$$
Hence, (2.1)-(2.2) has a unique solution

\[
A = \frac{\beta\eta(\eta + 1)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)y(s)
- \frac{\beta(T + 1)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1}(\eta - s)(\eta - s + 1)y(s),
\]

\[
B = \frac{2(1 - \beta\eta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)y(s)
+ \frac{\beta - \alpha}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1}(\eta - s)(\eta - s + 1)y(s).
\]

\[u(t) = \frac{\beta\eta(\eta + 1) + 2t(1 - \beta\eta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)y(s)
- \frac{\beta(T + 1) - (\beta - \alpha)t}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1}(\eta - s)(\eta - s + 1)y(s)
- \sum_{s=1}^{t-1}(t - s)y(s), \quad t \in \mathbb{N}_{T+1}.
\]

Lemma 2.2. Let \(0 < \alpha < (2T + 2)/\eta(\eta + 1), 0 < \beta < (2T + 2 - \alpha\eta(\eta + 1))/\eta(2T - \eta + 1)\). If \(y \in C(\mathbb{N}_{T+1}, [0, \infty))\) and \(y(t) \geq 0\) for \(t \in \mathbb{N}_{1,T}\), then the unique solution \(u\) of (2.1)-(2.2) satisfies \(u(t) \geq 0\) for \(t \in \mathbb{N}_{T+1}\).

Proof. From the fact that \(\Delta^2u(t - 1) = u(t + 1) - 2u(t) + u(t - 1) = -y(t) \leq 0\), we know that \(u(t) \geq (u(t + 1) + u(t - 1))/2\), so \(u(t + 1)/(t + 1) < u(t)/t\).

Hence

\[
\frac{u(T + 1) - u(0)}{T + 1} < \frac{u(\eta) - u(0)}{\eta + 1}, \quad \eta \in \mathbb{N}_{1,T},
\]

since \(u(T) \geq 0\) and \(u(0) \geq 0\) imply that \(u(t) \geq 0\) for \(t \in \mathbb{N}_{T+1}\).
Moreover, from $u(i) > (i/\eta)u(\eta) + ((\eta - i)/\eta)u(0)$, we get

$$
\sum_{s=1}^{\eta} u(s) > \left[ \frac{1}{\eta} u(\eta) + \frac{\eta - 1}{\eta} u(0) \right] + \left[ \frac{2}{\eta} u(\eta) + \frac{\eta - 2}{\eta} u(0) \right] + \cdots + \left[ \frac{\eta}{\eta} u(\eta) + \frac{\eta - \eta}{\eta} u(0) \right]
$$

$$
= \frac{1}{\eta} u(\eta) [1 + 2 + \cdots + \eta] + \frac{1}{\eta} u(0) [(\eta - 1) + (\eta - 2) + \cdots + 0]
$$

$$
= \frac{1}{\eta} u(\eta) \left[ \frac{1}{2} \eta (\eta + 1) \right] + \frac{1}{\eta} u(0) \left[ \eta^2 - \frac{1}{2} \eta (\eta + 1) \right]
$$

$$
= \frac{1}{2} \eta (\eta + 1) u(\eta) + \frac{1}{2} (\eta - 1) u(0).
$$

Combining (2.14) with (2.2), we can get

$$
u(0) > \frac{\beta(\eta + 1)}{2 - \beta(\eta - 1)} u(\eta), \quad (2.15)$$

again combining (2.2), (2.14), and (2.15), we obtain

$$u(T + 1) > \frac{\alpha(\eta + 1)}{2 - \beta(\eta - 1)} u(\eta), \quad (2.16)$$

such that

$$2 - \beta(\eta - 1) > 2 - \beta \eta > 2 - \frac{2T + 2 - \alpha \eta(\eta + 1)}{2T - \eta + 1} = \frac{2(T - \eta) + \alpha \eta(\eta + 1)}{2T - \eta + 1} > 0. \quad (2.17)$$

By using (2.13), (2.15), and (2.16), we obtain

$$\frac{2 - 2\beta \eta}{\eta} u(\eta) > \frac{(\alpha - \beta)(\eta + 1)}{T + 1} u(\eta). \quad (2.18)$$

If $u(0) < 0$, then $u(\eta) < 0$. It implies that $(2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1) \leq \beta$, a contradiction to $\beta < (2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1)$. If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \geq 0$, $u(T) \geq 0$, together with (2.13), we have

$$u(t) \geq 0, \quad t \in \mathbb{N}_{T+1}. \quad (2.19)$$

This proof is complete.

**Lemma 2.3.** Let $\alpha \eta(\eta + 1) > 2T + 2$, $\beta > \max \{(2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1), 0\}$. If $y \in C(\mathbb{N}_{T+1}, [0, \infty))$ and $y(t) \geq 0$ for $t \in \mathbb{N}_{1,T}$, then problem (2.1)-(2.2) has no positive solutions.

**Proof.** Suppose that problem (2.1)-(2.2) has a positive solution $u$ satisfying $u(t) \geq 0$, $t \in \mathbb{N}_{T+1}$, and there is a $\tau_0 \in \mathbb{N}_{1,T}$ such that $u(\tau_0) > 0$. 

Let \( u(T + 1) > 0 \), then \( \sum_{s=1}^{\eta} u(s) > 0 \). It implies

\[
u(0) = \beta \sum_{s=1}^{\eta} u(s) > \frac{2T + 2 - \alpha \eta(\eta + 1)}{\eta(2T - \eta + 1)} \sum_{s=1}^{\eta} u(s)
\geq \frac{\eta(T + 1)(u(0) + u(\eta)) - \eta(\eta + 1)u(T + 1)}{\eta(2T - \eta + 1)},
\]

that is,

\[
\frac{u(T + 1) - u(0)}{T + 1} > \frac{u(\eta) - u(0)}{\eta + 1},
\]

which is a contradiction to (2.13).

If \( u(T + 1) = 0 \), then \( \sum_{s=1}^{\eta} u(s)ds = 0 \). When \( \tau_0 \in \mathbb{N}_{1,\eta-1} \), we get \( u(\tau_0) > u(T) = 0 > u(\eta) \), which contradicts to (2.13). When \( \tau_0 \in \mathbb{N}_{\eta+1,T} \), we get \( u(\eta) \leq 0 = u(0) < u(\tau_0) \), which contradicts to (2.13) again. Therefore, no positive solutions exist.

Let \( E = C(\mathbb{N}_{T+1}, [0, \infty)) \), then \( E \) is a Banach space with respect to the norm

\[
\|u\| = \sup_{t \in \mathbb{N}_{T+1}} |u(t)|.
\]

**Lemma 2.4.** Let \( 0 < \alpha < (2T + 2)/\eta(\eta + 1) \), \( 0 < \beta < (2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1) \). If \( y \in C(\mathbb{N}_{T+1}, [0, \infty)) \) and \( y(t) \geq 0 \), then the unique solution to problem (2.1)-(2.2) satisfies

\[
\inf_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma \|u\|,
\]

where

\[
\gamma := \min \left\{ \frac{\alpha(\eta + 1)(T + 1 - \eta)}{(T + 1)(2 - \beta(\eta - 1)) - \alpha \eta(\eta + 1)(T + 1)'}, \frac{\alpha \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)'}, \frac{\beta(\eta + 1)(T + 1 - \eta)}{(2 - \beta(\eta - 1))(T + 1)'}, \frac{\beta \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)'} \right\}.
\]

**Proof.** Let \( u(t) \) be maximal at \( t = \tau_1 \), when \( \tau_1 \in \mathbb{N}_{1,T} \) and \( \|u\| = u(\tau_1) \). We divide the proof into two cases.
Case i. If \( u(0) \geq u(T + 1) \) and \( \inf_{t \leq \tau_1} u(t) = u(T + 1) \), then either \( 0 \leq \tau_1 \leq \eta < T + 1 \) or \( 0 < \eta < \tau_1 < T + 1 \), if \( 0 \leq \tau_1 \leq \eta < T + 1 \), then

\[
\begin{align*}
    u(\tau_1) & \leq u(T + 1) + \frac{u(T + 1) - u(\eta)}{T + 1 - \eta} (\tau_1 - (T + 1)) \\
        & \leq u(T + 1) + \frac{u(T + 1) - u(\eta)}{T + 1 - \eta} (0 - (T + 1)) \\
        & \leq u(T + 1) \left[ 1 - \left( \frac{(T + 1) - (T + 1)(2 - \beta(\eta - 1))/\alpha(\eta + 1)}{T + 1 - \eta} \right) \right] \\
        & \leq u(T + 1) \left[ \frac{(T + 1)(2 - \beta(\eta - 1)) - \alpha \eta(\eta + 1)}{\alpha(T + 1)(T + 1 - \eta)} \right].
\end{align*}
\]

This implies

\[
\inf_{t \leq \tau_1} u(t) \geq \frac{\alpha(T + 1)(T + 1 - \eta)}{(T + 1)(2 - \beta(\eta - 1)) - \alpha \eta(\eta + 1)} ||u||. \tag{2.26}
\]

Similarly, if \( 0 < \eta < \tau_1 < T + 1 \), from

\[
\frac{u(\eta)}{\eta} \geq \frac{u(\tau_1)}{\tau_1} \geq \frac{u(\tau_1)}{T + 1}, \tag{2.27}
\]

together with (2.16), we have

\[
u(T + 1) \geq \frac{\alpha \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)} u(\tau_1). \tag{2.28}\]

This implies

\[
\inf_{t \leq \tau_1} u(t) \geq \frac{\alpha \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)} ||u||. \tag{2.29}
\]

Case ii. If \( u(0) \leq u(T + 1) \) and \( \inf_{t \leq \tau_1} u(t) = u(0) \), then either \( 0 < \tau_1 < \eta < T + 1 \) or \( 0 < \eta \leq \tau_1 \leq T + 1 \), by (2.13). If \( 0 < \tau_1 < \eta < T + 1 \), from

\[
\frac{u(\eta)}{T + 1 - \eta} \geq \frac{u(\tau_1)}{T + 1 - \tau_1} \geq \frac{u(\tau_1)}{T + 1}, \tag{2.30}
\]

together with (2.15), we have

\[
u(0) > \frac{\beta(\eta + 1)(T + 1 - \eta)}{(2 - \beta(\eta - 1))(T + 1)} u(\tau_1). \tag{2.31}\]
Hence,

\[
\inf_{t \in \mathbb{N}_{T+1}} u(t) > \frac{\beta(\eta + 1)(T + 1 - \eta)}{(2 - \beta(\eta - 1))(T + 1)} \|u\|. \tag{2.32}
\]

If \(0 < \eta \leq \tau_1 \leq T + 1\), from

\[
\frac{u(\tau_1)}{T + 1} \leq \frac{u(\tau_1)}{\tau_1} \leq \frac{u(\eta)}{\eta}, \tag{2.33}
\]

together with \((2.15)\), we have

\[
u(0) < \frac{\beta \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)} u(\tau_1). \tag{2.34}\]

This implies

\[
\inf_{t \in \mathbb{N}_{T+1}} u(t) < \frac{\beta \eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)} \|u\|. \tag{2.35}
\]

This completes the proof.

\[\square\]

In the rest of the paper, we assume that \(0 < \alpha < (2T + 2)/\eta(\eta + 1)\), \(T \in \mathbb{N}_{1:T}\); \(0 < \beta < (2T + 2 - \alpha \eta(\eta + 1))/\eta(2T - \eta + 1)\). It is easy to see that the BVP \((1.2)\) has a solution \(u = u(t)\) if and only if \(u\) is a solution of the operator equation

\[
Au(t) \triangleq \frac{\beta \eta(\eta + 1) + 2t(1 - \beta \eta)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)u(s)f(u(s))
- \frac{\beta(T + 1) - (\beta - \alpha)t}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)u(s)f(u(s))
- \sum_{s=1}^{t-1} (t - s)u(s)f(u(s)). \tag{2.36}\]

Denote

\[
K = \left\{ u \in E : u \geq 0, \ \min_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma \|u\| \right\}, \tag{2.37}
\]

where \(\gamma\) is defined in \((2.24)\).
Proof. Superlinear Case

Let \( \frac{f}{u} \) hold. Since \( Au = u \) and from Lemmas 2.2 and 2.4, then \( A(K) \subseteq K \). It is also easy to check that \( A : K \rightarrow K \) is completely continuous. In the following, for the sake of convenience, set

\[
\Lambda_1 = \frac{(2T + 2)(1 - \beta \eta) + \beta \eta(\eta + 1)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)a(s), \\
\Lambda_2 = \frac{\gamma(2 - \beta \eta + \beta)(T - \eta)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}sa(s).
\]

(2.38)

3. Main Results

Now we are in the position to establish the main result.

**Theorem 3.1.** The BVP (1.2) has at least one positive solution in the case

(H1) \( f_0 = 0 \) and \( f_\infty = \infty \) (superlinear) or

(H2) \( f_0 = \infty \) and \( f_\infty = 0 \) (sublinear).

**Proof.** Superlinear Case

Let (H1) hold. Since \( f_0 = \lim_{u \to 0^+} \frac{(f(u))}{u} = 0 \) for any \( \varepsilon \in (0, \Lambda_1] \), there exists \( \rho_* \) such that

\[
f(u) \leq \varepsilon u \quad \text{for } u \in [0, \rho_*].
\]

(3.1)

Let \( \Omega_{\rho_*} = \{ u \in E : \|u\| < \rho_* \} \) for any \( u \in K \cap \partial \Omega_{\rho_*} \). From (3.1), we get

\[
Au(t) = \frac{2T(1 - \beta \eta) + \beta \eta(\eta + 1)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)a(s)f(u(s))
- \frac{\beta(T + 1) - (\beta - \alpha)t}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1}(T - s + 1)a(s)f(u(s))
- \sum_{s=1}^{t-1}(t - s)a(s)f(u(s))
\leq \frac{2T(1 - \beta \eta) + \beta \eta(\eta + 1)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)a(s)f(u(s))
\leq \frac{2(T + 1)(1 - \beta \eta) + \beta \eta(\eta + 1)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)a(s)f(u(s))
\leq \varepsilon \rho_* \frac{(2T + 2)(1 - \beta \eta) + \beta \eta(\eta + 1)}{(2T + 2 - \alpha \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T}(T - s + 1)a(s)
= \varepsilon \Lambda_1 \rho_* \leq \rho_* = \|u\|,
\]

where \( \varepsilon \) is a small positive constant.
which yields

$$\|Au\| \leq \|u\| \quad \text{for } u \in K \cap \partial \Omega_{\rho_*}. \quad (3.3)$$

Further, since $f_\infty = \lim_{u \to \infty} (f(u)/u) = \infty$, then, for any $M^* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho^* > \rho_*$ such that

$$f(u) \geq M^* u \quad \text{for } u \geq \gamma \rho^*. \quad (3.4)$$

Set $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$ for $u \in K \cap \partial \Omega_{\rho^*}$.

Since $u \in K$, $\min_{t \in N_i} u(t) \geq \gamma \|u\| = \gamma \rho^*$. Hence, for any $u \in K \cap \Omega_{\rho^*}$, from (3.4) and (2.23), we get

$$Au(\eta) = \frac{(2 - \beta \eta + \beta)\eta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))$$

$$- \frac{\beta(T + 1) - (\beta - \alpha)\eta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta} (\eta - s)(\eta - s + 1)a(s)f(u(s))$$

$$- \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s))$$

$$= \frac{(2 - \beta \eta + \beta)\eta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))$$

$$+ \frac{1}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)}$$

$$\times \sum_{s=1}^{\eta-1} (\eta - s)[- (2 - \beta \eta + \beta)T + (\beta(T - \eta) + a\eta + 1)s + (\eta - 1)\beta]a(s)f(u(s))$$

$$= \frac{(2 - \beta \eta + \beta)\eta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))$$

$$- \frac{T(2 - \beta \eta + \beta)}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s))$$

$$+ \frac{\beta(T - \eta) + a\eta + 1}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta s - s^2)a(s)f(u(s))$$

$$+ \frac{(\eta - 1)\beta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s))$$

$$\geq \frac{(2 - \beta \eta + \beta)\eta}{(2T + 2 - a\eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))$$
Let \( \Omega \) such that \( \rho \geq \beta \eta + \beta \) holds. In view of \( f \geq \beta \), for any \( u \in u - \beta \), we can get
\[
\rho \geq \gamma r \geq \gamma r s = M \Lambda r \rho
\]
\[
\Omega \in \{ T - \beta \eta \geq \gamma r \}
\]
which yields
\[
\| Au \| \geq \| u \| \quad \text{for } u \in K \cap \partial \Omega.
\]
Since \( f_\infty = \lim_{u \to \infty} (f(u)/u) = 0 \), then, for any \( \epsilon_1 \in (0, \Lambda_1^{-1}] \), there exists \( r_0 > r_* \) such that

\[
f(u) \leq \epsilon_1 u \quad \text{for} \quad u \in [r_0, \infty).
\] (3.10)

We have the following two cases.

Case i. Suppose that \( f(u) \) is unbounded, then, from \( f \in C([0, \infty), [0, \infty)) \), we know that there is \( r^* > r_0 \) such that

\[
f(u) \leq f(r^*) \quad \text{for} \quad u \in [0, r^*].
\] (3.11)

Since \( r^* > r_0 \), then, from (3.10) and (3.11), one has

\[
f(u) \leq f(r^*) \leq \epsilon_1 r^* \quad \text{for} \quad u \in [0, r^*].
\] (3.12)

For \( u \in K, \|u\| = r^* \), from (3.12), we obtain

\[
Au(t) \leq \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))
\leq \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)
\leq \epsilon_1 r^* \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)
= \epsilon_1 \Lambda_1 r^* \leq r^* = \|u\|.
\] (3.13)

Case ii. Suppose that \( f(u) \) is bounded, say \( f(u) \leq N \) for all \( u \in [0, \infty) \). Taking \( r^* \geq \max \{N/\epsilon_1, r_*\} \), for \( u \in K, \|u\| = r^* \), we have

\[
Au(t) \leq \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)f(u(s))
\leq N \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)
\leq \epsilon_1 r^* \frac{(2T + 2)(1 - \beta \eta) + \beta \eta (\eta + 1)}{(2T + 2 - a \eta(\eta + 1)) - \beta \eta(2T - \eta + 1)} \sum_{s=1}^{T} (T - s + 1)a(s)
= \epsilon_1 \Lambda_1 r^* \leq r^* = \|u\|.
\] (3.14)

Hence, in either case, we may always set \( \Omega_{r^*} = \{u \in E : \|u\| < r^*\} \) such that

\[
\|Au\| \leq \|u\| \quad \text{for} \quad u \in K \cap \partial \Omega_{r^*}.
\] (3.15)

Hence, from (3.9), (3.15), and Theorem 1.1, it follows that \( A \) has a fixed point in \( K \cap (\Omega_{r^*} \setminus \Omega_{r_*}) \) such that \( r_* \leq \|u\| \leq r^* \). The proof is complete. \( \square \)
4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

**Example 4.1.** Consider the BVP

\[
\Delta^2 u(t - 1) + t^2 u^k = 0, \quad t \in N_{1,4},
\]
\[
u(0) = \frac{1}{3} \sum_{s=1}^{2} u(s), \quad u(5) = \frac{2}{3} \sum_{s=1}^{2} u(s).
\]

Set \( \alpha = 2/3, \beta = 1/3, \eta = 2, T = 4, a(t) = t^2, \) and \( f(u) = u^k. \)

We can show that

\[
0 < \alpha = \frac{2}{3} < \frac{5}{3} = \frac{2T + 2}{\eta(\eta + 1)}, \quad 0 < \beta = \frac{1}{3} < \frac{3}{7} = \frac{2T + 2 - \alpha \eta (\eta + 1)}{\eta(2T - \eta + 1)}.
\]

**Case I.** \( k \in (1, \infty). \) In this case, \( f_0 = 0, f_\infty = \infty, \) and \( H_1 \) of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

**Case II.** \( k \in (0, 1). \) In this case, \( f_0 = \infty, f_\infty = 0, \) and \( H_2 \) of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

**Example 4.2.** Consider the BVP

\[
\Delta^2 u(t - 1) + e^t t^e \left( \frac{\pi \sin u + 2 \cos u}{u^2} \right) = 0, \quad t \in N_{1,4},
\]
\[
u(0) = \frac{1}{4} \sum_{s=1}^{3} u(s), \quad u(5) = \frac{1}{3} \sum_{s=1}^{3} u(s).
\]

Set \( \alpha = 1/3, \beta = 1/4, \eta = 3, T = 4, a(t) = e^t t^e, \) \( f(u) = (\pi \sin u + 2 \cos u)/u^2. \)

We can show that

\[
0 < \alpha = \frac{1}{3} < \frac{5}{6} = \frac{2T + 2}{\eta(\eta + 1)}, \quad 0 < \beta = \frac{1}{4} < \frac{3}{7} = \frac{2T + 2 - \alpha \eta (\eta + 1)}{\eta(2T - \eta + 1)}.
\]

Through a simple calculation we can get \( f_0 = \infty, f_\infty = 0. \) Thus, by \( H_2 \) of Theorem 3.1, we can get BVP (4.3) has at least one positive solution.

**Acknowledgment**

This research is supported by the Centre of Excellence in Mathematics, Commission on Higher Education, Thailand.
References
