Research Article

Sharp Bounds by the Generalized Logarithmic Mean for the Geometric Weighted Mean of the Geometric and Harmonic Means

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We present sharp upper and lower generalized logarithmic mean bounds for the geometric weighted mean of the geometric and harmonic means.

1. Introduction

For \( p \in \mathbb{R} \) the generalized logarithmic mean \( L_p(a, b) \) of two positive numbers \( a \) and \( b \) is defined by

\[
L_p(a, b) = \begin{cases} 
  a, & a = b, \\
  \left[ \frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)} \right]^{1/p}, & p \neq 0, p \neq -1, a \neq b, \\
  \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}, & p = 0, a \neq b, \\
  \frac{b-a}{\log b - \log a}, & p = -1, a \neq b.
\end{cases}
\]  

(1.1)

It is well-known that \( L_p(a, b) \) is continuous and strictly increasing with respect to \( p \in \mathbb{R} \) for fixed \( a \) and \( b \) with \( a \neq b \). In the recent past, the generalized logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for \( L_p \) can be
found in the literature [1–23]. The generalized logarithmic mean has applications in convex function, economics, physics, and even in meteorology [24–27]. In [26] the authors study a variant of Jensen’s functional equation involving $L_p$, which appear in a heat conduction problem. Let $A(a, b) = (a+b)/2$, $I(a, b) = (1/e)(b^p/a^s)^{1/(b-a)}$, $L(a, b) = (b-a)/(\log b - \log a)$, $G(a, b) = \sqrt[3]{ab}$, and $H(a, b) = 2ab/(a + b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers $a$ and $b$ with $a \neq b$, respectively. Then it is well known that

$$\min\{a, b\} < H(a, b) < G(a, b) = L_{-2}(a, b) < L(a, b) = L_{-1}(a, b)$$

$$< I(a, b) = L_0(a, b) < A(a, b) = L_1(a, b) < \max\{a, b\}. \quad (1.2)$$

In [28–30], the authors present bounds for $L$ and $I$ in terms of $G$ and $A$.

**Proposition 1.1.** For all positive real numbers $a$ and $b$ with $a \neq b$, one has

$$A^{1/3}(a, b)G^{2/3}(a, b) < L(a, b) < \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b),$$

$$\frac{1}{3}G(a, b) + \frac{2}{3}A(a, b) < I(a, b). \quad (1.3)$$

The proof of the following Proposition 1.2 can be found in [31].

**Proposition 1.2.** For all positive real numbers $a$ and $b$ with $a \neq b$, we have

$$\sqrt{G(a, b)A(a, b)} < \sqrt{L(a, b)I(a, b)} < \frac{1}{2}(L(a, b) + I(a, b)) < \frac{1}{2}(G(a, b) + A(a, b)). \quad (1.4)$$

For $r \in \mathbb{R}$ the $r$th power mean $M_r(a, b)$ of two positive numbers $a$ and $b$ is defined by

$$M_r(a, b) = \begin{cases} \left(\frac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt[3]{ab}, & r = 0. \end{cases} \quad (1.5)$$

The main properties of these means are given in [32]. Several authors discussed the relationship of certain means to $M_r$. The following sharp bounds for $L$, $I$, $(IL)^{1/2}$, and $(I + L)/2$ in terms of power means are proved in [31, 33–37].

**Proposition 1.3.** For all positive real numbers $a$ and $b$ with $a \neq b$ one has

$$M_0(a, b) < L(a, b) < M_{1/3}(a, b), \quad M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b),\$$

$$M_0(a, b) < I^{1/2}(a, b)L^{1/2}(a, b) < M_{1/2}(a, b),$$

$$\frac{1}{2}[I(a, b) + L(a, b)] < M_{1/2}(a, b). \quad (1.6)$$

The following three results were established by Alzer and Qiu in [38].
Theorem 2.1. For \( \alpha \in (0, 1) \) and all \( a, b > 0 \), one has the following:

1. \( L_{3\alpha - 5}(a, b) = G^a(a, b)H^{1-a}(a, b) = L_{-2/(3\alpha)}(a, b) \) for \( \alpha = 2/3 \),
2. \( L_{3\alpha - 5}(a, b) \geq G^a(a, b)H^{1-a}(a, b) \geq L_{-(2/\alpha)}(a, b) \) for \( 0 < \alpha < 2/3 \), and \( L_{3\alpha - 5}(a, b) \leq G^a(a, b)H^{1-a}(a, b) \leq L_{-(2/\alpha)}(a, b) \) for \( 2/3 < \alpha < 1 \), with equality if and only if \( a = b \), and the parameters \( 3\alpha - 5 \) and \(-2/\alpha\) in each inequality cannot be improved.

Proof. (1) If \( \alpha = 2/3 \) and \( a = b \), then (1.1) implies that \( L_{3\alpha - 5}(a, b) = G^a(a, b)H^{1-a}(a, b) = L_{-(2/\alpha)}(a, b) = a \).
If $\alpha = 2/3$ and $a \neq b$, then (1.1) leads to

$$L_{3\alpha-5}(a,b) = L_{-(2/\alpha)}(a,b) = L_{-3}(a,b) = \left[\frac{a^{-2} - b^{-2}}{2(b - a)}\right]^{-1/3}$$  \hspace{1cm} (2.1)

$$= (ab)^{1/3} \left(\frac{2ab}{a + b}\right)^{1/3} = G^{2/3}(a,b)H^{1/3}(a,b) = G^*(a,b)H^{1-\alpha}(a,b).$$

(2) If $a = b$, then from (1.1) we clearly see that $L_{3\alpha-5}(a,b) = G^*(a,b)H^{1-\alpha}(a,b) = L_{-(2/\alpha)}(a,b) = a$ for any $\alpha \in (0,1)$.

If $a \neq b$, without loss of generality, we assume $a > b$. Let $a/b = t > 1$ and

$$f(t) = \log L_{3\alpha-5}(a,b) - \log\left[G^*(a,b)H^{1-\alpha}(a,b)\right].$$  \hspace{1cm} (2.2)

Then (1.1) and simple computations yield

$$f(t) = \frac{1}{3\alpha - 5} \log \frac{t^{3\alpha-4} - 1}{(3\alpha - 4)(t - 1)} - \frac{\alpha}{2} \log t - (1 - \alpha) \log \frac{2t}{1 + t},$$

$$\lim_{t \to 1^-} f(t) = 0,$$

$$f'(t) = \frac{t^{4-3\alpha}}{t(t^2 - 1)(t^{4-3\alpha} - 1)} g(t),$$  \hspace{1cm} (2.4)

where $g(t) = (2 - \alpha/2)t^{3\alpha-2} - ((2 - \alpha)(2 - 3\alpha)/5 - 3\alpha)t^{3\alpha-3} + ((1 - \alpha)(2 - 3\alpha)/2(5 - 3\alpha)t^{3\alpha-4} - ((1 - \alpha)(2 - 3\alpha)/2(5 - 3\alpha)t^2 + ((2 - \alpha)(2 - 3\alpha)/(5 - 3\alpha)t - (2 - \alpha)/2,$

$$g(1) = 0,$$

$$g'(t) = \frac{(2 - \alpha)(3\alpha - 2)}{2} t^{3\alpha-3} - \frac{3(2 - \alpha)(2 - 3\alpha)(\alpha - 1)}{5 - 3\alpha} t^{3\alpha-4}$$

$$+ \frac{(1 - \alpha)(2 - 3\alpha)(3\alpha - 4)}{2(5 - 3\alpha)} t^{3\alpha-5} - \frac{(1 - \alpha)(2 - 3\alpha)}{5 - 3\alpha} t$$

$$+ \frac{(2 - \alpha)(2 - 3\alpha)}{(5 - 3\alpha)},$$  \hspace{1cm} (2.5)

$$g''(1) = 0,$$

$$g''(t) = \frac{3(2 - \alpha)(3\alpha - 2)(\alpha - 1)}{2} t^{3\alpha-4} - \frac{3(2 - \alpha)(2 - 3\alpha)(\alpha - 1)(3\alpha - 4)}{5 - 3\alpha} t^{3\alpha-5}$$

$$- \frac{(1 - \alpha)(2 - 3\alpha)(3\alpha - 4)}{2} t^{3\alpha-6} - \frac{(1 - \alpha)(2 - 3\alpha)}{(5 - 3\alpha)} t$$

$$+ \frac{(2 - \alpha)(4 - 3\alpha)(3\alpha - 2)}{(5 - 3\alpha)} t^2,$$  \hspace{1cm} (2.6)

$$g'''(1) = 0,$$

$$g'''(t) = \frac{3}{2} (1 - \alpha)(2 - a)(4 - 3\alpha)(3\alpha - 2) t^{3\alpha-7} (t - 1)^2.$$  \hspace{1cm} (2.7)
If $0 < \alpha < 2/3$, then (2.7) implies

$$g''(t) < 0$$  \hspace{1cm} (2.8)$$

for $t > 1$.

From (2.3)–(2.6) and (2.8) we know that $f(t) > 0$ for $t > 1$.

If $2/3 < \alpha < 1$, then (2.7) leads to

$$g''(t) > 0$$  \hspace{1cm} (2.9)$$

for $t > 1$. Therefore $f(t) < 0$ for $t > 1$ follows from (2.3)–(2.6) and (2.9).

Let

$$h(t) = \log L_{-(2/\alpha)}(a, b) - \log \left[ G^\alpha(a, b) H^{1-\alpha}(a, b) \right]$$  \hspace{1cm} (2.10)$$

for $t = a/b > 1$; then (1.1) and elementary calculations lead to

$$h(t) = -\frac{\alpha}{2} \log \left( \frac{t(2-\alpha)/(\alpha-1)}{(t^2-1)(t^{2-\alpha}/\alpha - 1)} \right) - \frac{\alpha}{2} \log t - (1-\alpha) \log \frac{2t}{1+t},$$  \hspace{1cm} (2.11)$$

$$h'(t) = -\frac{t^{(2-\alpha)/\alpha}}{t(t^2-1)(t^{2-\alpha}/\alpha - 1)} \nu(t),$$  \hspace{1cm} (2.12)$$

where

$$\nu(t) = (2-\alpha)/2 t^{(3\alpha-2)/\alpha} + (3\alpha-2)/2 t^{(2\alpha-2)/\alpha} - (3\alpha-2)/2 t - (2-\alpha)/2,$$

$$\nu(1) = 0,$$

$$\nu'(t) = \frac{(2-\alpha)(3\alpha-2)/\alpha}{2\alpha} t^{(2\alpha-2)/\alpha} + \frac{(3\alpha-2)(\alpha-1)}{\alpha} t^{(2-\alpha)/\alpha} - \frac{3\alpha-2}{2},$$  \hspace{1cm} (2.13)$$

$$\nu'(1) = 0,$$

$$\nu''(t) = \frac{(2-\alpha)(1-\alpha)(2-3\alpha)}{\alpha} t^{-2/\alpha} (t-1).$$  \hspace{1cm} (2.14)$$

If $\alpha \in (0, 2/3)$, then (2.15) implies

$$\nu''(t) > 0$$  \hspace{1cm} (2.16)$$

for $t > 1$.

From (2.11)–(2.14) and (2.16) we know that $h(t) < 0$ for $t > 1$.

If $\alpha \in (2/3, 1)$, then (2.15) leads to

$$\nu''(t) < 0$$  \hspace{1cm} (2.17)$$

for $t > 1$. Therefore, $h(t) > 0$ for $t > 1$ follows from (2.11)–(2.14) and (2.17).
Next, we prove that the parameters \(-2/\alpha\) and \(3\alpha - 5\) in either case cannot be improved. The proof is divided into two cases.

**Case 1 (\(\alpha \in (0, 2/3)\)).** For any \(\epsilon > 0\) and \(x \in (0, 1)\), from (1.1) one has

\[
\left[ G(1, 1 + x)H^{1-\alpha}(1, 1 + x) \right]^{5-3\alpha+\epsilon} - \left[ L_{3\alpha-5+\epsilon}(1, 1 + x) \right]^{5-3\alpha+\epsilon}
= \frac{f_1(x)}{(1 + x/2)^{(1-\alpha)(5-3\alpha+\epsilon)} ((1 + x)4-3\alpha+\epsilon - 1)}, \tag{2.18}
\]

where \(f_1(x) = (1 + x)(1-\alpha)(5-3\alpha+\epsilon) \left[ (1 + x)4-3\alpha+\epsilon - 1 \right] - (4 - 3\alpha + \epsilon)x(1 + x)^{-3\alpha+\epsilon} \left( 1 + x/2 \right)^{(1-\alpha)(5-3\alpha+\epsilon)}. \)

Let \(x \to 0\); making use of the Taylor expansion, we get

\[
f_1(x) = \frac{\epsilon(4 - 3\alpha + \epsilon)(5 - 3\alpha + \epsilon)}{24} x^3 + o(x^3). \tag{2.19}
\]

Equations (2.18) and (2.19) imply that for any \(\alpha \in (0, 2/3)\) and \(\epsilon > 0\) there exists \(\delta = \delta(\epsilon, \alpha) \in (0, 1)\), such that \(L_{3\alpha-5+\epsilon}(1, 1 + x) < G(1, 1 + x)H^{1-\alpha}(1, 1 + x)\) for \(x \in (0, \delta)\).

On the other hand, for any \(\epsilon \in (0, (2/\alpha) - 1)\) we have

\[
L_{-2/(\alpha + \epsilon)}(1, t) - G(1, t)H^{1-\alpha}(1, t)
= t^{\alpha/(2-\epsilon)} \left\{ \left[ \frac{1 - t^{2/\alpha + \epsilon + 1}}{(2\alpha - \epsilon - 1)(1 - 1/t)} \right] - t^{-\epsilon/2(2-\epsilon)} (\frac{2t}{1 + t})^{1-\alpha} \right\},
\]

\[
\lim_{t \to +\infty} \left\{ \left[ \frac{1 - t^{2/\alpha + \epsilon + 1}}{(2\alpha - \epsilon - 1)(1 - 1/t)} \right] - t^{-\epsilon/2(2-\epsilon)} (\frac{2t}{1 + t})^{1-\alpha} \right\} = \left( \frac{2}{\alpha} - \epsilon - 1 \right)^{\alpha/(2-\epsilon)} > 0. \tag{2.20}
\]

From (2.20) we know that for any \(\alpha \in (0, 2/3)\) and \(\epsilon \in (0, (2/\alpha) - 1)\) there exists \(T = T(\epsilon, \alpha) > 1\), such that \(L_{-2/(\alpha + \epsilon)}(1, t) > G(1, t)H^{1-\alpha}(1, t)\) for \(t \in (T, \infty)\).

**Case 2 (\(\alpha \in (2/3, 1)\)).** For any \(\epsilon \in (0, 4 - 3\alpha)\) and \(x \in (0, 1)\), from (1.1) one has

\[
\left[ L_{3\alpha-5+\epsilon}(1, 1 + x) \right]^{5-3\alpha-\epsilon} - \left[ G(1, 1 + x)H^{1-\alpha}(1, 1 + x) \right]^{5-3\alpha-\epsilon}
= \frac{f_2(x)}{(1 + x/2)^{(1-\alpha)(5-3\alpha-\epsilon)} ((1 + x)4-3\alpha-\epsilon - 1)}, \tag{2.21}
\]

where \(f_2(x) = (4 - 3\alpha - \epsilon)x(1 + x)^{4-3\alpha-\epsilon} \left( 1 + x/2 \right)^{(1-\alpha)(5-3\alpha-\epsilon) - (1 + x)(1-\alpha/2)(5-3\alpha-\epsilon) - (1 + x)(4-3\alpha-\epsilon - 1)}. \)

Let \(x \to 0\); making use of the Taylor expansion, we have

\[
f_2(x) = \frac{\epsilon(4 - 3\alpha - \epsilon)(5 - 3\alpha - \epsilon)}{24} x^3 + o(x^3). \tag{2.22}
\]
Equations (2.21) and (2.22) imply that for any $\alpha \in (2/3, 1)$ and $e \in (0, 4 - 3\alpha)$ there exists $\delta = \delta(e, \alpha) \in (0, 1)$, such that $L_{3a-5}((1 + x) > G^a((1 + x)H^{1-a}(1 + x)$ for $x \in (0, \delta)$. On the other hand, for any $e > 0$, we have

$$G^a(1, t)H^{1-a}(1, t) - L_{-(2/\alpha) - e}(1, t)$$

$$= \frac{\alpha}{2} \left\{ \begin{array}{l}
\frac{2t}{1 + t} \left( \frac{1 + \alpha}{\alpha} \right) - t^{-e\alpha^2/2(2+e\alpha)} \left[ \frac{1 - t^{-(2/\alpha + e - 1)}}{(2/\alpha + e - 1)(1 - 1/t)} \right]^{-\alpha/(2+e\alpha)} \\
\frac{2t}{1 + t} - t^{-e\alpha^2/2(2+e\alpha)} \left[ \frac{1 - t^{-(2/\alpha + e - 1)}}{(2/\alpha + e - 1)(1 - 1/t)} \right]^{-\alpha/(2+e\alpha)} \end{array} \right\} 
$$

$$\lim_{t \to +\alpha} \left\{ \begin{array}{l}
\frac{2t}{1 + t} \left( \frac{1 + \alpha}{\alpha} \right) - t^{-e\alpha^2/2(2+e\alpha)} \left[ \frac{1 - t^{-(2/\alpha + e - 1)}}{(2/\alpha + e - 1)(1 - 1/t)} \right]^{-\alpha/(2+e\alpha)} \\
\frac{2t}{1 + t} - t^{-e\alpha^2/2(2+e\alpha)} \left[ \frac{1 - t^{-(2/\alpha + e - 1)}}{(2/\alpha + e - 1)(1 - 1/t)} \right]^{-\alpha/(2+e\alpha)} \end{array} \right\} = 2^{1-a} > 0. \tag{2.23}$$

From (2.23) we know that for any $\alpha \in (2/3, 1)$ and $e > 0$ there exists $T = T(e, \alpha) > 1$, such that $L_{-(2/\alpha) - e}(1, t) < G^a(1, t)H^{1-a}(1, t)$ for $t \in (T, \infty)$.

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**References**


