Research Article
An Iterative Algorithm for a Hierarchical Problem

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A general hierarchical problem has been considered, and an explicit algorithm has been presented for solving this hierarchical problem. Also, it is shown that the suggested algorithm converges strongly to a solution of the hierarchical problem.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. The hierarchical problem is of finding $\tilde{x} \in \text{Fix}(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T), \quad (1.1)$$

where $S, T$ are two nonexpansive mappings and $\text{Fix}(T)$ is the set of fixed points of $T$. Recently, this problem has been studied by many authors (see, e.g., [1–15]). The main reason is that this problem is closely associated with some monotone variational inequalities and convex programming problems (see [16–19]).

Now, we briefly recall some historic results which relate to the problem (1.1).

For solving the problem (1.1), in 2006, Moudafi and Mainge [1] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1 - s)[tS(x_{t,s}) + (1 - t)T(x_{t,s})] \quad (1.2)$$
and proved that the net \( \{x_t,s\} \) defined by (1.2) strongly converges to \( x_t \) as \( s \rightarrow 0 \), where \( x_t \) satisfies \( x_t = \text{proj}_{\text{Fix}(P)} Q(x_t) \), where \( P : C \rightarrow C \) is a mapping defined by

\[
P_t(x) = tS(x) + (1 - t)T(x), \quad \forall x \in C, \ t \in (0,1),
\]

or, equivalently, \( x_t \) is the unique solution of the quasivariational inequality

\[
0 \in (I - Q)x_t + N_{\text{Fix}(P)}(x_t),
\]

where the normal cone to \( \text{Fix}(P) \), \( N_{\text{Fix}(P)} \), is defined as follows:

\[
N_{\text{Fix}(P)} : x \mapsto \begin{cases} 
\{ u \in H : \langle y - x, u \rangle \leq 0 \}, & \text{if } x \in \text{Fix}(P), \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

Moreover, as \( t \rightarrow 0 \), the net \( \{x_t\} \) in turn weakly converges to the unique solution \( x_\infty \) of the fixed point equation \( x_\infty = \text{proj}_{\Omega} Q(x_\infty) \) or, equivalently, \( x_\infty \) is the unique solution of the variational inequality

\[
0 \in (I - Q)x_\infty + N_\Omega(x_\infty).
\]

Recently, Moudafi [2] constructed an explicit iterative algorithm:

\[
x_{n+1} = (1 - \delta_n)x_n + \delta_n(\sigma_nSx_n + (1 - \sigma_n)Tx_n), \quad \forall n \geq 0,
\]

where \( \{\delta_n\} \) and \( \{\sigma_n\} \) are two real numbers in \( (0,1) \). By using this iterative algorithm, Moudafi [2] only proved a weak convergence theorem for solving the problem (1.1).

In order to obtain a strong convergence result, Mainge and Moudafi [3] further introduced the following iterative algorithm:

\[
x_{n+1} = (1 - \delta_n)Qx_n + \delta_n[\sigma_nSx_n + (1 - \sigma_n)Tx_n], \quad \forall n \geq 0,
\]

where \( \{\delta_n\} \) and \( \{\sigma_n\} \) are two real numbers in \( (0,1) \), and proved that, under appropriate conditions, the iterative sequence \( \{x_n\} \) generated by (1.8) has strong convergence.

Subsequently, some authors have studied some algorithms on hierarchical fixed problems (see, e.g., [4–15]).

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding \( \bar{x} \in \text{Fix}(T) \) such that, for any \( n \geq 1 \),

\[
\langle W_n\bar{x} - \bar{x}, x - \bar{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T),
\]

where \( W_n \) is the \( W \)-mapping defined by (2.3) below and \( T \) is a nonexpansive mapping, and introduce an explicit iterative algorithm which converges strongly to a solution \( \bar{x} \) of the hierarchical problem (1.9).
2. Preliminaries

Let $C$ a nonempty closed convex subset of a real Hilbert space $H$. Recall that a mapping $Q : C \to C$ is said to be contractive if there exists a constant $\gamma \in (0, 1)$ such that

$$
\|Qx - Qy\| \leq \gamma \|x - y\|, \quad \forall x, y \in C.
$$

A mapping $T : C \to C$ is called nonexpansive if

$$
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.
$$

Forward, we use $\text{Fix}(T)$ to denote the fixed points set of $T$.

Let $\{T_i\}_{i=1}^\infty : C \to C$ be an infinite family of nonexpansive mappings and $\{\xi_i\}_{i=1}^\infty$ a real number sequence such that $0 \leq \xi_i \leq 1$ for each $i \geq 1$.

For each $n \geq 1$, define a mapping $W_n : C \to C$ as follows:

$$
\begin{align*}
U_{n, n+1} &= I, \\
U_{n, n} &= \xi_n T_n U_{n, n+1} + (1 - \xi_n)I, \\
U_{n, n-1} &= \xi_{n-1} T_{n-1} U_{n, n} + (1 - \xi_{n-1})I, \\
&\vdots \\
U_{n, k} &= \xi_k T_k U_{n, k+1} + (1 - \xi_k)I, \\
U_{n, k-1} &= \xi_{k-1} T_{k-1} U_{n, k} + (1 - \xi_{k-1})I, \\
&\vdots \\
U_{n, 2} &= \xi_2 T_2 U_{n, 3} + (1 - \xi_2)I, \\
W_n &= U_{n, 1} = \xi_1 T_1 U_{n, 2} + (1 - \xi_1)I.
\end{align*}
$$

Such $W_n$ is called the $W$-mapping generated by $\{T_i\}_{i=1}^\infty$ and $\{\xi_i\}_{i=1}^\infty$.

**Lemma 2.1** (see [20]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive mappings of $C$ into itself with $\bigcap_{n=1}^\infty \text{Fix}(T_n) \neq \emptyset$. Let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq b < 1$ for each $i \geq 1$. Then one has the following results:

1. For any $x \in C$ and $k \geq 1$, the limit $\lim_{n \to \infty} U_{n, k} x$ exists;
2. $\text{Fix}(W) = \bigcap_{n=1}^\infty \text{Fix}(T_n)$.

Using Lemma 3.1 in [21], we can define a mapping $W$ of $C$ into itself by $Wx = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n, 1} x$ for all $x \in C$. Thus we have the following.

**Lemma 2.2** (see [21]). If $\{x_n\}$ is a bounded sequence in $C$, then one has

$$
\lim_{n \to \infty} \|W x_n - W_n x_n\| = 0.
$$
Lemma 2.3 (see [22]). Let $C$ be a nonempty closed convex of a real Hilbert space $H$ and $T : C \to C$ be a nonexpansive mapping. Then $T$ is demiclosed on $C$, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \rightharpoonup 0$, then $x = Tx$.

Lemma 2.4 (see [23]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n + \eta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}, \{\eta_n\}$ are two sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n \to \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$;
(iii) $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

3. Main Results

In this section, we introduce our algorithm and give its convergence analysis.

Algorithm 3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $\{T_n\}_{n=1}^{\infty}$ be infinite family of nonexpansive mappings of $C$ into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0, 1)$. For any $x_0 \in C$, let $\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n W_n x_n + (1 - \alpha_n) T(\beta_n Q x_n + (1 - \beta_n) x_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$ and $W_n$ is the $W$-mapping defined by (2.3).

Now, we give the convergence analysis of the algorithm.

Theorem 3.2. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of $C$ into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0, 1)$. Assume that the set $\Omega$ of solutions of the hierarchical problem (1.9) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0, 1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} (\beta_n / \alpha_n) = 0$;
(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\lim_{n \to \infty} (1 / \beta_n) |(1 / \alpha_n) - (1 / \alpha_{n-1})| = 0$ and $\lim_{n \to \infty} (\prod_{i=1}^{n-1} \xi_i / \alpha_n \beta_n) = \lim_{n \to \infty} (1 / \alpha_n) |1 - (\beta_n / \beta_{n-1})| = 0$.

Then $\lim_{n \to \infty} (\|x_n + x_n\| / \alpha_n) = 0$ and every weak cluster point of the sequence $\{x_n\}$ solves the following variational inequality

$$\bar{x} \in \Omega,$$

$$\langle (I - Q) \bar{x}, x - \bar{x} \rangle \geq 0, \quad \forall x \in \Omega.$$
Proof. Set \( y_n = \beta_n Qx_n + (1 - \beta_n)x_n \) for each \( n \geq 0 \). Then we have

\[
y_n - y_{n-1} = \beta_n Qx_n + (1 - \beta_n)x_n - \beta_n Qx_{n-1} - (1 - \beta_n)x_{n-1} \\
= \beta_n(Qx_n - Qx_{n-1}) + (\beta_n - \beta_{n-1})Qx_{n-1} + (1 - \beta_n)(x_n - x_{n-1}) \\
+ (\beta_{n-1} - \beta_n)x_{n-1}.
\]

(3.3)

It follows that

\[
\|y_n - y_{n-1}\| \leq \gamma\beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|) \\
= [1 - (1 - \gamma)\beta_n]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|).
\]

(3.4)

From (3.1), we have

\[
x_n - x_{n-1} = \alpha_n W_n x_n + (1 - \alpha_n)Ty_{n-1} - \alpha_n W_{n-1} x_{n-1} - (1 - \alpha_n)Ty_{n-1} \\
= \alpha_n(W_n x_n - W_{n-1} x_{n-1}) + (\alpha_n - \alpha_{n-1})W_n x_{n-1} + \alpha_{n-1}(W_{n-1} x_{n-1} - W_n x_{n-1}) \\
+ (1 - \alpha_n)(Ty_{n-1} - Ty_{n-1}) + (\alpha_{n-1} - \alpha_n)Ty_{n-1}.
\]

(3.5)

Then we obtain

\[
\|x_n - x_{n-1}\| \leq \alpha_n\|W_n x_n - W_{n-1} x_{n-1}\| + (1 - \alpha_n)\|Ty_{n-1} - Ty_{n-1}\| \\
+ |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}\|W_n x_{n-1} - W_{n-1} x_{n-1}\| \\
\leq \alpha_n\|x_n - x_{n-1}\| + (1 - \alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) \\
+ \alpha_{n-1}\|W_n x_{n-1} - W_{n-1} x_{n-1}\|.
\]

(3.6)

From (2.3), since \( T_i \) and \( U_{i,j} \) are nonexpansive, we have

\[
\|W_n x_{n-1} - W_{n-1} x_{n-1}\| = \|\xi_1 T_1 U_{n,2} x_{n-1} - \xi_1 T_1 U_{n-1,2} x_{n-1}\| \\
\leq \|\xi_1 U_{n,2} x_{n-1} - U_{n-1,2} x_{n-1}\| \\
= \|\xi_2 T_2 U_{n,3} x_{n-1} - \xi_2 T_2 U_{n-1,3} x_{n-1}\| \\
\leq \|\xi_3 T_3 U_{n,4} x_{n-1} - \xi_3 T_3 U_{n-1,4} x_{n-1}\| \\
\leq \cdots \\
\leq \|\xi_{n-1} T_{n-1} U_{n,n} x_{n-1} - \xi_{n-1} T_{n-1} U_{n-1,n} x_{n-1}\| \\
\leq M_1 \prod_{i=1}^{n-1} \xi_i.
\]

(3.7)
where $M_1$ is a constant such that $\sup_{n \geq 1} \{\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\|\} \leq M_1$. Substituting (3.4) and (3.7) into (3.6), we get

$$
\|x_{n+1} - x_n\| \leq \alpha_n\|x_n - x_{n-1}\| + (1 - \alpha_n) [1 - (1 - \gamma)\beta_n] \|x_n - x_{n-1}\| \\
+ |\beta_n - \beta_{n-1}| (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
+ |\alpha_n - \alpha_{n-1}| (\|W_nx_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}M_1 \prod_{i=1}^{n-1} \xi_i \\
= [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \|x_n - x_{n-1}\| \\
+ |\beta_n - \beta_{n-1}| (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
+ |\alpha_n - \alpha_{n-1}| (\|W_nx_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}M_1 \prod_{i=1}^{n-1} \xi_i.
$$

(3.8)

Therefore, it follows that

$$
\frac{\|x_{n+1} - x_n\|}{\alpha_n} \leq [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_n} \\
+ \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
+ \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} (\|W_nx_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}M_1 \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \\
= [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
+ [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \left( \frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \right) \\
+ \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} (\|Qx_{n-1}\| + \|x_{n-1}\|) \\
+ \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} (\|W_nx_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}M_1 \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \\
\leq [1 - (1 - \gamma)\beta_n(1 - \alpha_n)] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
+ \left( \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n} \right) M
$$
\[
\begin{aligned}
&= \left[ 1 - (1 - \gamma) \beta_n (1 - \alpha_n) \right] \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + (1 - \gamma) \beta_n (1 - \alpha_n) \\
\times &\left\{ \frac{M}{(1 - \gamma)(1 - \alpha_n)} \left( \frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} \\
&\quad + \frac{1}{\beta_n} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n \beta_n} \right) \right\},
\end{aligned}
\]
(3.9)

where \(M\) is a constant such that

\[
\sup_{n \geq 1} \left\{ M, \|x_n - x_{n-1}\|, (\|W_n x_{n-1}\| + \|T y_{n-1}\|), (\|Q x_{n-1}\| + \|x_{n-1}\|) \right\} \leq M.
\]
(3.10)

From (iii), we note that \(\lim_{n \to \infty} (1/\alpha_{n-1}) |\alpha_n - \alpha_{n-1}|/\beta_n \alpha_n = 0\), which implies that

\[
\lim_{n \to \infty} \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.
\]
(3.11)

Thus it follows from (iii) and (3.11) that

\[
\lim_{n \to \infty} \left( \frac{1}{\beta_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| + \frac{1}{\beta_n} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{1}{\beta_n} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} + \frac{\prod_{i=1}^{n-1} \xi_i}{\alpha_n \beta_n} \right) = 0.
\]
(3.12)

Hence, applying Lemma 2.4 to (3.9), we immediately conclude that

\[
\lim_{n \to \infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n} = 0.
\]
(3.13)

This implies that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]
(3.14)

Thus, from (3.1) and (3.14), we have

\[
\lim_{n \to \infty} \|x_n - Ty_n\| = 0.
\]
(3.15)

At the same time, we note that

\[
y_n - x_n = \beta_n (Q x_n - x_n) \longrightarrow 0.
\]
(3.16)

Hence we get

\[
\lim_{n \to \infty} \|y_n - Ty_n\| = 0.
\]
(3.17)
Since the sequence \( \{x_n\} \) is bounded, \( \{y_n\} \) is also bounded. Thus there exists a subsequence of \( \{y_n\} \), which is still denoted by \( \{y_n\} \) which converges weakly to a point \( \bar{x} \in H \). Therefore, \( \bar{x} \in \text{Fix}(T) \) by (3.17) and Lemma 2.3. By (3.1), we observe that
\[
x_{n+1} - x_n = \alpha_n (W_n x_n - x_n) + (1 - \alpha_n) (T y_n - y_n) + (1 - \alpha_n) \beta_n (Q x_n - x_n),
\]
that is,
\[
\frac{x_n - x_{n+1}}{\alpha_n} = (I - W_n) x_n + \frac{1 - \alpha_n}{\alpha_n} (I - T) y_n + \frac{\beta_n (1 - \alpha_n)}{\alpha_n} (I - Q) x_n.
\]
Set \( z_n = (x_n - x_{n+1})/\alpha_n \) for each \( n \geq 1 \), that is,
\[
z_n = (I - W_n) x_n + \frac{1 - \alpha_n}{\alpha_n} (I - T) y_n + \frac{\beta_n (1 - \alpha_n)}{\alpha_n} (I - Q) x_n.
\]
Using monotonicity of \( I - T \) and \( I - W_n \), we derive that, for all \( u \in \text{Fix}(T) \),
\[
\langle z_n, x_n - u \rangle \\
= \langle (I - W_n) x_n, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle (I - T) y_n, y_n - u \rangle \\
+ \frac{1 - \alpha_n}{\alpha_n} \langle (I - T) y_n, x_n - y_n \rangle + \frac{\beta_n (1 - \alpha_n)}{\alpha_n} \langle (I - Q) x_n, x_n - y_n \rangle \\
\geq \langle (I - W_n) u, x_n - u \rangle + \frac{\beta_n (1 - \alpha_n)}{\alpha_n} \langle (I - Q) x_n, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle (I - T) y_n, x_n - Q x_n \rangle \\
= \langle (I - W) u, x_n - u \rangle + \langle (W - W_n) u, x_n - u \rangle + \frac{\beta_n (1 - \alpha_n)}{\alpha_n} \langle (I - Q) x_n, x_n - u \rangle \\
+ \frac{1 - \alpha_n}{\alpha_n} \langle (I - T) y_n, x_n - Q x_n \rangle.
\]
But, since \( z_n \to 0 \), \( \beta_n/\alpha_n \to 0 \) and \( \lim_{n \to \infty} \|W_n u - W u\| = 0 \) (by Lemma 2.2), it follows from the above inequality that
\[
\limsup_{n \to \infty} \langle (I - W) u, x_n - u \rangle \leq 0, \quad \forall u \in \text{Fix}(T).
\]
This suffices to guarantee that \( \omega_w(x_n) \subset \Omega \). As a matter of fact, if we take any \( x^* \in \omega_w(x_n) \), then there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( x_{n_j} \to x^* \). Therefore, we have
\[
\langle (I - W) u, x^* - u \rangle = \lim_{j \to \infty} \left( \langle (I - W) u, x_{n_j} - u \rangle \right) \leq 0, \quad \forall u \in \text{Fix}(T).
\]
Theorem 3.3. Let $x^* \in \text{Fix}(T)$. Hence $x^*$ solves the following problem:

$$\begin{align*}
x^* & \in \text{Fix}(T), \\
\langle (I - W)u, x^* - u \rangle & \leq 0, \quad \forall u \in \text{Fix}(T).
\end{align*}$$

(3.24)

It is obvious that this is equivalent to the problem (1.9) since $W_n \to W$ uniformly in any bounded set (by Lemma 2.2). Thus $x^* \in \Omega$.

Let $\bar{x}$ be the unique solution of the variational inequality (3.2). Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle (I - Q)\bar{x}, x_n - \bar{x} \rangle = \lim_{i \to \infty} \langle (I - Q)\bar{x}, x_{n_i} - \bar{x} \rangle. $$

(3.25)

Without loss of generality, we may further assume that $x_{n_i} \to \bar{x}$. Then $\bar{x} \in \Omega$. Therefore, we have

$$\limsup_{n \to \infty} \langle (I - Q)\bar{x}, x_n - \bar{x} \rangle = \langle (I - Q)\bar{x}, \bar{x} - \bar{x} \rangle \geq 0. $$

(3.26)

This completes the proof. \hfill \Box

Theorem 3.3. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_n\}_{n=1}^\infty$ be infinite family of nonexpansive mappings of $C$ into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. Assume that the set $\Omega$ of solutions of the hierarchical problem (1.9) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0,1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \beta_n/\alpha_n = 0$ and $\lim_{n \to \infty} \alpha_n^2/\beta_n = 0$;

(ii) $\sum_{n=0}^\infty \beta_n = \infty$;

(iii) $\lim_{n \to \infty} (1/\beta_n)[1/\alpha_n - (1/\alpha_{n-1})] = 0$ and $\lim_{n \to \infty} \prod_{i=1}^{n-1} (\beta_i/\alpha_i) \alpha_n \beta_n = \lim_{n \to \infty} (1/\alpha_n)[1 - (\beta_{n-1}/\beta_n)] = 0$;

(iv) there exists a constant $k > 0$ such that $\|x - Tx\| \geq k\text{Dist}(x, \text{Fix}(T))$, where

$$\text{Dist}(x, \text{Fix}(T)) = \inf_{y \in \text{Fix}(T)} \|x - y\|. $$

(3.27)

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\bar{x} \in \text{Fix}(T)$, which solves the variational inequality problem (3.2).

Proof. From (3.1), we have

$$x_{n+1} - \bar{x} = \alpha_n(W_n x_n - W_n \bar{x}) + \alpha_n(W_n \bar{x} - \bar{x}) + (1 - \alpha_n)(T y_n - \bar{x}).$$

(3.28)
Thus we have

\[ \| x_{n+1} - \bar{x} \|^2 \leq \| \alpha_n (W_n x_n - W_n \bar{x}) + (1 - \alpha_n) (T y_n - \bar{x}) \|^2 + 2 \alpha_n \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \]
\[ \leq (1 - \alpha_n) \| T y_n - \bar{x} \|^2 + 2 \alpha_n \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \]
\[ \leq (1 - \alpha_n) \| y_n - \bar{x} \|^2 + 2 \alpha_n \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \] (3.29)

At the same time, we observe that

\[ \| y_n - \bar{x} \|^2 = \| (1 - \beta_n) (x_n - \bar{x}) + \beta_n (Q x_n - Q \bar{x}) + \beta_n (Q \bar{x} - \bar{x}) \|^2 \]
\[ \leq \| (1 - \beta_n) (x_n - \bar{x}) + \beta_n (Q x_n - Q \bar{x}) \|^2 + 2 \beta_n \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle \]
\[ \leq (1 - \beta_n) \| x_n - \bar{x} \|^2 + \beta_n \| Q x_n - Q \bar{x} \|^2 + 2 \beta_n \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle \] (3.30)
\[ \leq (1 - \beta_n) \| x_n - \bar{x} \|^2 + \beta_n \gamma^2 \| x_n - \bar{x} \|^2 + 2 \beta_n \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle \]
\[ = \left[ 1 - (1 - \gamma^2) \beta_n \right] \| x_n - \bar{x} \|^2 + 2 \beta_n \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle. \]

Substituting (3.30) into (3.29), we get

\[ \| x_{n+1} - \bar{x} \|^2 \leq \alpha_n \| x_n - \bar{x} \|^2 + (1 - \alpha_n) \left[ 1 - (1 - \gamma^2) \beta_n \right] \| x_n - \bar{x} \|^2 \]
\[ + 2 \beta_n (1 - \alpha_n) \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle + 2 \alpha_n \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \]
\[ = \left[ 1 - (1 - \gamma^2) \beta_n (1 - \alpha_n) \right] \| x_n - \bar{x} \|^2 + 2 \beta_n (1 - \alpha_n) \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle \]
\[ + 2 \alpha_n \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \]
\[ = \left[ 1 - (1 - \gamma^2) \beta_n (1 - \alpha_n) \right] \| x_n - \bar{x} \|^2 + (1 - \gamma^2) \beta_n (1 - \alpha_n) \]
\[ \times \left\{ \frac{2}{1 - \gamma^2} \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle + \frac{2}{(1 - \gamma^2)(1 - \alpha_n)} \times \frac{\alpha_n}{\beta_n} \langle W_n \bar{x} - \bar{x}, x_{n+1} - \bar{x} \rangle \right\}. \] (3.31)

By Theorem 3.2, we note that every weak cluster point of the sequence \( \{ x_n \} \) is in \( \Omega \). Since \( y_n - x_n \to 0 \), then every weak cluster point of \( \{ y_n \} \) is also in \( \Omega \). Consequently, since \( \bar{x} = \text{proj}_\Omega (Q \bar{x}) \), we easily have

\[ \limsup_{n \to \infty} \langle Q \bar{x} - \bar{x}, y_n - \bar{x} \rangle \leq 0. \] (3.32)
Thus it follows that

\[ \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle = \langle W_n\tilde{x} - \tilde{x}, \text{proj}_{\text{Fix}(T)} x_{n+1} - \tilde{x} \rangle + \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1} \rangle. \]  

(3.33)

Since \( \tilde{x} \) is a solution of the problem (1.9) and \( \text{proj}_{\text{Fix}(T)} x_{n+1} \in \text{Fix}(T) \), we have

\[ \langle W_n\tilde{x} - \tilde{x}, \text{proj}_{\text{Fix}(T)} x_{n+1} - \tilde{x} \rangle \leq 0. \]  

(3.34)

Thus it follows that

\[ \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \leq \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1} \rangle \]

\[ \leq \|W_n\tilde{x} - \tilde{x}\| \|x_{n+1} - \text{proj}_{\text{Fix}(T)} x_{n+1}\| \]

\[ = \|W_n\tilde{x} - \tilde{x}\| \times \text{Dist}(x_{n+1}, \text{Fix}(T)) \]

\[ \leq \frac{1}{k} \|W_n\tilde{x} - \tilde{x}\| \|x_{n+1} - Tx_{n+1}\|. \]  

(3.35)

We note that

\[ \|x_{n+1} - Tx_{n+1}\| \leq \|x_{n+1} - Tx_n\| + \|Tx_n - Tx_{n+1}\| \]

\[ \leq \alpha_n \|W_n x_n - Tx_n\| + (1 - \alpha_n) \|Ty_n - Tx_n\| + \|x_{n+1} - x_n\| \]

\[ \leq \alpha_n \|W_n x_n - Tx_n\| + \|y_n - x_n\| + \|x_{n+1} - x_n\| \]

\[ \leq \alpha_n \|W_n x_n - Tx_n\| + \beta_n \|Qx_n - x_n\| + \|x_{n+1} - x_n\|. \]  

(3.36)

Hence we have

\[ \frac{\alpha_n}{\beta_n} \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \]

\[ \leq \frac{\alpha_n^2}{\beta_n} \left( \frac{1}{k} \|W_n\tilde{x} - \tilde{x}\| \|W_n x_n - Tx_n\| \right) + \alpha_n \left( \frac{1}{k} \|W_n\tilde{x} - \tilde{x}\| \|Qx_n - x_n\| \right) \]

\[ + \frac{\alpha_n^2}{\beta_n} \|x_{n+1} - x_n\| \left( \frac{1}{k} \|W_n\tilde{x} - \tilde{x}\| \right). \]  

(3.37)

From Theorem 3.2, we have \( \lim_{n \to \infty} \|x_{n+1} - x_n\| / \alpha_n = 0 \). At the same time, we note that \( \{(1/k)\|W_n\tilde{x} - \tilde{x}\|\|W_n x_n - Tx_n\|\}, \{(1/k)\|W_n\tilde{x} - \tilde{x}\|\|Qx_n - x_n\|\} \), and \( \{(1/k)\|W_n\tilde{x} - \tilde{x}\|\} \) are all bounded. Hence it follows from (i) and the above inequality that

\[ \limsup_{n \to \infty} \frac{\alpha_n}{\beta_n} \langle W_n\tilde{x} - \tilde{x}, x_{n+1} - \tilde{x} \rangle \leq 0. \]  

(3.38)
Finally, by (3.31)–(3.38) and Lemma 2.4, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in \text{Fix}(T)$. This completes the proof.

**Remark 3.4.** In the present paper, we consider the hierarchical problem (1.9) which includes the hierarchical problem (1.1) as a special case.

From the above discussion, we can easily deduce the following result.

**Algorithm 3.5.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $S$ a nonexpansive mapping of $C$ into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0, 1)$. For any $x_0 \in C$, let $\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n Sx_n + (1 - \alpha_n)T(\beta_n Qx_n + (1 - \beta_n)x_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0, 1)$.

**Corollary 3.6.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonexpansive mapping. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0, 1)$. Assume that the set $\Omega$ of solutions of the hierarchical problem (1.1) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0, 1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and

(i) $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \beta_n / \alpha_n = 0$ and $\lim_{n \to \infty} \alpha_n^2 / \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n \to \infty} (1 / \beta_n)(1 / \alpha_n) - (1 / \alpha_{n-1}) = 0$ and $\lim_{n \to \infty} (1 / \alpha_n)\left[1 - (\beta_{n-1} / \beta_n)\right] = 0$;

(iv) there exists a constant $k > 0$ such that $\|x - Tx\| \geq k \text{Dist}(x, \text{Fix}(T))$, where

$$\text{Dist}(x, \text{Fix}(T)) = \inf_{y \in \text{Fix}(T)} \|x - y\|.$$  

Then the sequence $\{x_n\}$ defined by (3.39) converges strongly to a point $\bar{x} \in \text{Fix}(T)$, which solves the hierarchical problem (1.1).

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**References**


