Research Article

Implicit Mann Type Iteration Method
Involving Strictly Hemiconttractive Mappings
in Banach Spaces

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We proved that the modified implicit Mann iteration process can be applied to approximate the fixed point of strictly hemicontractive mappings in smooth Banach spaces.

1. Introduction

Let \( K \) be a nonempty subset of an arbitrary Banach space \( X \) and let \( X^* \) be its dual space. The symbols \( D(T) \) and \( F(T) \) stand for the domain and the set of fixed points of \( T \) (for a single-valued mapping \( T : X \to X, \ x \in X \) is called a fixed point of \( T \) iff \( Tx = x \)). We denote by \( J \) the normalized duality mapping from \( X \) to \( 2^{X^*} \) defined by

\[
J(x) = \left\{ f^* \in X^* : \left\langle x, f^* \right\rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad x \in X,
\]

(1.1)

where \( \left\langle \cdot, \cdot \right\rangle \) denotes the duality pairing. In a smooth Banach space, \( J \) is singlevalued (we denoted by \( j \)).

Remark 1.1. (1) \( X \) is called uniformly smooth if \( X^* \) is uniformly convex.

(2) In a uniformly smooth Banach space, \( J \) is uniformly continuous on bounded subsets of \( X \).

Let \( T : D(T) \subset X \to X \) be a mapping.
Definition 1.2. The mapping $T$ is called *Lipschitz* if there exists a constant $L > 0$ such that

$$
\|Tx - Ty\| \leq L\|x - y\|,
$$

for all $x, y \in D(T)$. If $L = 1$, then $T$ is called *nonexpansive* and if $0 \leq L < 1$, then $T$ is called *contractive*.

Definition 1.3 (see [1, 2]). (1) The mapping $T$ is said to be *pseudocontractive* if

$$
\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|,
$$

for all $x, y \in D(T)$ and $r > 0$.

(2) The mapping $T$ is said to be *strongly pseudocontractive* if there exists a constant $t > 1$ such that

$$
\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|,
$$

for all $x, y \in D(T)$ and $r > 0$.

(3) The mapping $T$ is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists a constant $t > 1$ such that

$$
\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|,
$$

for all $y \in D(T)$ and $r > 0$.

(4) The mapping $T$ is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a constant $t > 1$ such that

$$
\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|,
$$

for all $x \in D(T), q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive mapping is local strongly pseudocontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of $T$ in case $T$ is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of $L_p$ (or $l_p$) into itself. Schu [3] generalized the result in [1] to both uniformly continuous strongly pseudocontractive mappings and real smooth Banach spaces. Park [4] extended the result in [1] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [5] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Afterwards, several generalizations have been made in various directions (see, e.g., [6–13]).
In 2001, Xu and Ori [14] introduced the following implicit iteration process for a finite family of nonexpansive mappings \( \{T_i : i \in I\} \) (here \( I = \{1, 2, \ldots, N\} \)) with \( \{\alpha_n\} \) a real sequence in \((0, 1)\) and an initial point \( x_0 \in K \):

\[
\begin{align*}
  x_1 &= (1 - \alpha_1)x_0 + \alpha_1T_1x_1, \\
  x_2 &= (1 - \alpha_2)x_1 + \alpha_2T_2x_2, \\
  &\vdots \\
  x_N &= (1 - \alpha_N)x_{N-1} + \alpha_NT_Nx_N, \\
  x_{N+1} &= (1 - \alpha_{N+1})x_N + \alpha_{N+1}T_{N+1}x_{N+1}, \\
  &\vdots
\end{align*}
\]

which can be written in the following compact form:

\[
x_n = (1 - \alpha_n)x_{n-1} + \alpha_nT_nx_n, \quad n \geq 1, \tag{1.7}
\]

where \( T_n = T_{n(\text{mod} \ N)} \) (here the mod \( N \) function takes values in \( I \)). Xu and Ori [14] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters \( \{\alpha_n\} \) are sufficient to guarantee the strong convergence of the sequence \( \{x_n\} \).

In [11], Osilike proved the following results.

**Theorem 1.4.** Let \( X \) be a real Banach space and let \( K \) be a nonempty closed convex subset of \( X \). Let \( \{T_i : i \in I\} \) be \( N \) strictly pseudocontractive mappings from \( K \) to \( K \) with \( \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Let \( \{\alpha_n\} \) be a real sequence satisfying the following conditions:

(i) \( 0 < \alpha_n < 1 \),
(ii) \( \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty \),
(iii) \( \sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty \).

From arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) by the implicit iteration process (1.8). Then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( \{T_i : i \in I\} \) if and only if

\[
\lim \inf_{n \to \infty} d(x_n, \mathcal{F}) = 0.
\]

**Remark 1.5.** One can easily see that for \( \alpha_n = 1 - 1/n^{1/2} \), \( \sum_{n=1}^{\infty} (1 - \alpha_n)^2 = \infty \). Hence the results of Osilike [11] are needed to be improved.

Let \( K \) be a nonempty closed bounded convex subset of an arbitrary smooth Banach space \( X \) and let \( T : K \to K \) be a continuous strictly hemicontractive mapping. We proved that the implicit Mann type iteration method converges strongly to a unique fixed point of \( T \).

The results presented in this paper extend and improve the corresponding results particularly in [1, 3, 4, 7, 8, 10, 11, 13, 15].

### 2. Preliminaries

We need the following results.
Lemma 2.1 (see [4]). Let $X$ be a smooth Banach space. Suppose that one of the following holds:

(a) $J$ is uniformly continuous on any bounded subsets of $X$,
(b) $(x - y, j(x) - j(y)) \leq \|x - y\|^2$ for all $x, y$ in $X$,
(c) for any bounded subset $D$ of $X$, there is a function $c : [0, \infty) \to [0, \infty)$ such that

$$\text{Re}(x - y, j(x) - j(y)) \leq c(\|x - y\|),$$

(2.1)

for all $x, y \in D$, where $c$ satisfies $\lim_{t \to 0^+} (c(t)/t) = 0$.

Then for any $\epsilon > 0$ and any bounded subset $K$, there exists $\delta > 0$ such that

$$\|sx + (1-s)y\|^2 \leq (1 - 2s)\|y\|^2 + 2s \text{Re}(x, j(y)) + 2s\epsilon,$$

(2.2)

for all $x, y \in K$ and $s \in [0, \delta]$.

Remark 2.2. (1) If $X$ is uniformly smooth, then (a) in Lemma 2.1 holds.
(2) If $X$ is a Hilbert space, then (b) in Lemma 2.1 holds.

Lemma 2.3 (see [8]). Let $T : D(T) \subset X \to X$ be a mapping with $F(T) \neq \emptyset$. Then $T$ is strictly hemicontractive if and only if there exists a constant $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x - q) \in J(x - q)$ satisfying

$$\text{Re} (x - Tx, j(x - q)) \geq \left(1 - \frac{1}{t}\right)\|x - q\|^2.$$  

(2.3)

Lemma 2.4 (see [10]). Let $X$ be an arbitrary normed linear space and let $T : D(T) \subset X \to X$ be a mapping.

(a) If $T$ is a local strongly pseudocontractive mapping and $F(T) \neq \emptyset$, then $F(T)$ is a singleton and $T$ is strictly hemicontractive.

(b) If $T$ is strictly hemicontractive, then $F(T)$ is a singleton.

Lemma 2.5 (see [10]). Let $\{\theta_n\}, \{\sigma_n\}$, and $\{\omega_n\}$ be nonnegative real sequences and let $\epsilon' > 0$ be a constant satisfying

$$\sigma_{n+1} \leq (1 - \theta_n)\sigma_n + \epsilon'\theta_n + \omega_n, \quad n \geq 1,$$

(2.4)

where $\sum_{n=1}^{\infty} \theta_n = \infty$, $\theta_n \leq 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \omega_n < \infty$. Then $\lim \sup_{n \to \infty} \sigma_n \leq \epsilon'$.

3. Main Results

We now prove our main results.

Lemma 3.1. Let $X$ be a smooth Banach space. Suppose that one of the following holds:

(a) $J$ is uniformly continuous on any bounded subsets of $X$,
Lemma 3.1. Let \(x - y, j(x) - j(y)\) \(\leq \|x - y\|^2\) for all \(x, y\) in \(X\),

(c) for any bounded subset \(D\) of \(X\), there is a function \(c : [0, \infty) \to [0, \infty)\) such that

\[
\Re(x - y, j(x) - j(y)) \leq c(\|x - y\|)
\]

for all \(x, y \in D\), where \(c\) satisfies \(\lim_{t \to 0} c(t)/t = 0\).

Then for any \(e > 0\) and any bounded subset \(K\), there exists \(\delta > 0\) such that

\[
\|ax + \beta y + \gamma z\|^2 \leq (1 - 2\alpha)\|x\|^2 + 2\frac{\alpha\beta}{1 - \alpha} \Re(y, j(x)) + 2e\alpha
\]

for all \(x, y, z \in K\) and \(\alpha, \beta, \gamma \in [0, \delta]; \alpha + \beta + \gamma = 1\).

Proof. For \(\alpha, \beta, \gamma \in [0, \delta]; \alpha + \beta + \gamma = 1\), by using (2.2), consider

\[
\|ax + \beta y + \gamma z\|^2 = \left\|ax + (1 - \alpha) \left(\frac{\beta}{1 - \alpha} y + \frac{\gamma}{1 - \alpha} z\right)\right\|^2
\]

\[
\leq (1 - 2\alpha)\|x\|^2 + 2e\alpha + 2\alpha \Re\left(\frac{\beta}{1 - \alpha} y + \frac{\gamma}{1 - \alpha} z, j(x)\right)
\]

\[
= (1 - 2\alpha)\|x\|^2 + 2e\alpha + 2\frac{\alpha\beta}{1 - \alpha} \Re(y, j(x)) + 2\frac{\alpha\gamma}{1 - \alpha} \Re(z, j(x)).
\]

This completes the proof. \(\square\)

Theorem 3.2. Let \(X\) be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let \(K\) be a nonempty closed bounded convex subset of \(X\) and let \(T : K \to K\) be a continuous strictly hemicontractive mapping. Let \(\{\alpha_n\}, \{\beta_n\}\) and \(\{\gamma_n\}\) be real sequences in [0, 1] satisfying conditions

(iv) \(\alpha_n + \beta_n + \gamma_n = 1\), for all \(n \geq 1\),

(v) \(\sum_{n=1}^{\infty} \alpha_n = \infty\),

(vi) \(\sum_{n=1}^{\infty} \beta_n < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\).

For a sequence \(\{v_n\}\) in \(K\), suppose that \(\{x_n\}\) is the sequence generated from an arbitrary \(x_0 \in K\) by

\[
x_n = \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T v_n, \quad n \geq 1,
\]

satisfying \(\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty\).

Then the sequence \(\{x_n\}\) converges strongly to a unique fixed point \(q\) of \(T\).

Proof. By [2, Corollary 1], \(T\) has a unique fixed point \(q\) in \(K\). It follows from Lemma 2.4 that \(F(T)\) is a singleton. That is, \(F(T) = \{q\}\) for some \(q \in K\).
Set $M = 1 + \text{diam } K$. It is easy to verify that

$$M = \sup_{n \geq 1} \|x_n - q\| + \sup_{n \geq 1} \|Tx_n - q\| + \sup_{n \geq 1} \|Tv_n - q\|. \tag{3.5}$$

Also

$$\|v_n - q\|^2 \leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2\|v_n - x_n\|\|x_n - q\|$$

$$\leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2M\|v_n - x_n\|. \tag{3.6}$$

Consider

$$\|x_n - q\|^2 = \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2$$

$$= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2$$

$$\leq \alpha_n \|x_{n-1} - q\|^2$$

$$+ \beta_n \|Tx_n - q\|^2 + \gamma_n \|Tv_n - q\|^2$$

$$\leq \|x_{n-1} - q\|^2 + M^2(\beta_n + \gamma_n), \tag{3.7}$$

where the first inequality holds by the convexity of $\| \cdot \|^2$.

Now we put $k = 1/t$, where $t$ satisfies (2.3). Using (3.4) and Lemma 3.1, we infer that

$$\|x_n - q\|^2 = \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2$$

$$= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2$$

$$\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} \text{Re} \langle Tx_n - q, j(x_{n-1} - q) \rangle$$

$$+ 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} \text{Re} \langle Tv_n - q, j(x_{n-1} - q) \rangle + 2 \epsilon \alpha_n$$

$$= (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} \text{Re} \langle Tx_n - q, j(x_{n-1} - q) \rangle$$

$$+ 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} \text{Re} \langle Tx_n - q, j(x_{n-1} - q) \rangle$$

$$+ 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} \text{Re} \langle Tv_n - q, j(v_n - q) \rangle$$

$$+ 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} \text{Re} \langle Tv_n - q, j(x_{n-1} - q) \rangle + 2 \epsilon \alpha_n$$

$$\leq (1 - 2\alpha_n) \|x_{n-1} - q\|^2 + 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} \|x_n - q\|^2$$

$$+ 2 \frac{\alpha_n \beta_n}{1 - \alpha_n} \|Tx_n - q\| \|j(x_{n-1} - q) - j(x_n - q)\|$$

$$+ 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} k \|v_n - q\|^2$$

$$+ 2 \frac{\alpha_n \gamma_n}{1 - \alpha_n} \|Tv_n - q\| \|j(x_{n-1} - q) - j(v_n - q)\| + 2 \epsilon \alpha_n$$
\begin{equation}
\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} k \|x_n - q\|^2 + 2M \frac{\alpha_n\beta_n}{1 - \alpha_n} \delta_n \\
+ 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} k \|v_n - q\|^2 + 2M \frac{\alpha_n\gamma_n}{1 - \alpha_n} \eta_n + 2\epsilon \alpha_n \\
\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1 - \alpha_n} k \|x_n - q\|^2 \\
+ 2\frac{\alpha_n\gamma_n}{1 - \alpha_n} k \|v_n - q\|^2 + 2M \alpha_n \max\{\delta_n, \eta_n\} + 2\epsilon \alpha_n,
\end{equation}

where

\begin{align}
\delta_n &= \|j(x_{n-1} - q) - j(x_n - q)\|, \\
\eta_n &= \|j(x_{n-1} - q) - j(v_n - q)\|.
\end{align}

Also, we have

\begin{align}
\|x_{n-1} - x_n\| &= \|x_{n-1} - \alpha_n x_n - \beta_n T x_n - \gamma_n T v_n\| \\
&= \|\beta_n (x_{n-1} - T x_n) + \gamma_n (x_{n-1} - T v_n)\| \\
&\leq \beta_n \|x_{n-1} - T x_n\| + \gamma_n \|x_{n-1} - T v_n\| \\
&\leq 2M (\beta_n + \gamma_n) \\
&< \infty
\end{align}

implies

\begin{equation}
\|x_{n-1} - x_n\| \to 0,
\end{equation}

as \( n \to \infty \), and consequently

\begin{equation}
\|x_{n-1} - v_n\| \leq \|x_{n-1} - x_n\| + \|x_n - v_n\| \to 0
\end{equation}

as \( n \to \infty \). Since \( J \) is uniformly continuous on any bounded subsets of \( X \), we have

\begin{equation}
\delta_n, \eta_n \to 0 \text{ as } n \to \infty.
\end{equation}

For any given \( \epsilon > 0 \) and the bounded subset \( K \), there exists a \( \delta > 0 \) satisfying (2.2). Note that (3.13) and (vi) ensure that there exists an \( N \) such that

\begin{equation}
\beta_n, \gamma_n < \min\left\{\delta, \frac{\epsilon}{8M^2 k}\right\}, \quad \delta_n, \eta_n \leq \frac{\epsilon}{4M}, \quad n \geq N.
\end{equation}
Now substituting (3.6) in (3.8) to obtain
\[
\|x_n - q\|^2 \leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2k\alpha_n\|x_n - q\|^2 \\
+ 2M\alpha_n\max\{\delta_n, \eta_n\} + 2\varepsilon\alpha_n \\
+ 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right),
\]  
(3.15)
by using (3.7), implies
\[
\|x_n - q\|^2 \leq (1 - 2(1 - k)\alpha_n)\|x_{n-1} - q\|^2 + 2\varepsilon\alpha_n \\
+ 2M^2k\alpha_n(\beta_n + \gamma_n) + 2M\alpha_n\max\{\delta_n, \eta_n\} \\
+ 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right) \\
\leq (1 - 2(1 - k)\alpha_n)\|x_{n-1} - q\|^2 + 3\varepsilon\alpha_n \\
+ 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right),
\]  
(3.16)
for all \(n \geq N\).

Put
\[
\sigma_n = \|x_{n-1} - q\|^2, \quad \theta_n = 2(1 - k)\alpha_n, \quad \varepsilon' = \frac{3\varepsilon}{2(1 - k)}, \\
\omega_n = 2\frac{\alpha_n\gamma_n}{1 - \alpha_n}k\left(\|v_n - x_n\|^2 + 2M\|v_n - x_n\|\right),
\]  
(3.17)
and we have from (3.16)
\[
\sigma_{n+1} \leq (1 - \theta_n)\sigma_n + \varepsilon'\theta_n + \omega_n, \quad n \geq 1.
\]  
(3.18)

For \(k < 1/2\), set \(\delta = 1/2(1 - k) < 1\). Because \(\alpha_n \leq \delta\), we imply \(1 - \alpha_n \geq 1 - \delta \) and \(2(1 - k)\alpha_n \leq 1\). Now observe that \(\sum_{n=1}^{\infty} \theta_n = \infty\),\(\theta_n \leq 1\) for all \(n \geq 1\) and \(\sum_{n=1}^{\infty} \omega_n < \infty\). It follows from Lemma 2.5 that

\[
\limsup_{n \to \infty} \|x_n - q\|^2 \leq \varepsilon'.
\]  
(3.19)

Letting \(\varepsilon' \to 0^+\), we obtain that \(\limsup_{n \to \infty} \|x_n - q\|^2 = 0\), which implies that \(x_n \to q\) as \(n \to \infty\). This completes the proof. \(\square\)

**Corollary 3.3.** Let \(X\) be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let \(K\) be a nonempty closed bounded convex subset of \(X\) and let \(T : K \to K\) be a Lipschitz strictly hemicontractive mapping. Let \(\{\alpha_n\}, \{\beta_n\}\) and \(\{\gamma_n\}\) be real sequences in \([0, 1]\) satisfying the conditions (iv)–(vi).

From arbitrary \(x_0 \in K\), define the sequence \(\{x_n\}\) by the implicit iteration process (3.4). Then the sequence \(\{x_n\}\) converges strongly to a unique fixed point \(q\) of \(T\).
Corollary 3.4. Let $X$ be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let $K$ be a nonempty closed bounded convex subset of $X$ and let $T : K \to K$ be a continuous strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the conditions (v) and $\lim_{n \to \infty} \alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point $q$ of $T$.

Corollary 3.5. Let $X$ be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let $K$ be a nonempty closed bounded convex subset of $X$ and let $T : K \to K$ be a Lipschitz strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the conditions (v) and $\lim_{n \to \infty} \alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). Then the sequence $\{x_n\}$ converges strongly to a unique fixed point $q$ of $T$.

Remark 3.6. Similar results can be found for the iteration processes involved error terms; we omit the details.

Remark 3.7. Theorem 3.2 and Corollary 3.3 extend and improve Theorem 1.4 in the following directions.

We do not need the assumption $\lim \inf_{n \to \infty} d(x_n, F) = 0$ as in Theorem 1.4.

4. Applications for Multistep Implicit Iterations

Let $K$ be a nonempty closed convex subset of a smooth Banach space $X$ and let $T_1, T_2, \ldots, T_p : K \to K (p \geq 2)$ be a family of $p + 1$ mappings.

Algorithm 4.1. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the implicit iteration process of arbitrary fixed order $p \geq 2$:

$$
\begin{align*}
x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y^1_n, \\
y'_n &= \beta_n x_{n-1} + \left(1 - \beta_n^i\right) T_{i+1} y_{i+1}^i, \quad i = 1, 2, \ldots, p - 2, \\
y_{p-1}^n &= \beta_n^{p-1} x_{n-1} + \left(1 - \beta_n^{p-1}\right) T_p x_n, \quad n \geq 1,
\end{align*}
$$

which is called the multistep implicit iteration process, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\beta_n^i\}, i = 1, 2, \ldots, p - 1$ are real sequences in $[0, 1]$ and $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$.

For $p = 3$, we obtain the following three-step implicit iteration process.

Algorithm 4.2. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$
\begin{align*}
x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y^1_n, \\
y^1_n &= \beta_n^{1} x_{n-1} + \left(1 - \beta_n^{1}\right) T_2 y^2_n, \\
y^2_n &= \beta_n^{2} x_{n-1} + \left(1 - \beta_n^{2}\right) T_3 x_n, \quad n \geq 1,
\end{align*}
$$
where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \}, \{ \beta_n^1 \} \) and \( \{ \beta_n^2 \} \) are real sequences in \([0, 1]\) satisfying some certain conditions.

For \( p = 2 \), we obtain the following two-step implicit iteration process.

**Algorithm 4.3.** For a given \( x_0 \in K \), compute the sequence \( \{ x_n \} \) by the iteration process

\[
x_n = \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\
y_n^1 = \beta_n^1 x_{n-1} + \left( 1 - \beta_n^1 \right) T_2 x_n, \quad n \geq 1,
\]

where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) and \( \{ \beta_n^1 \} \) are real sequences in \([0, 1]\) satisfying some certain conditions.

If \( T_1 = T, T_2 = I \) and \( \beta_n^1 = 0 \) in (4.3), we obtain the following implicit Mann iteration process.

**Algorithm 4.4.** For any given \( x_0 \in K \), compute the sequence \( \{ x_n \} \) by the iteration process

\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_n, \quad n \geq 1,
\]

where \( \{ \alpha_n \} \) is a real sequence in \([0, 1]\) satisfying some certain conditions.

**Theorem 4.5.** Let \( X \) be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let \( K \) be a nonempty closed bounded convex subset of \( X \) and let \( T, T_1, T_2, \ldots, T_p : K \to K \) be \( p + 1 \) mappings. Let \( T, T_1 \) be continuous strictly hemicontractive mappings. Let \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) and \( \{ \beta_n^i \}, i = 1, 2, \ldots, p - 1 \) be real sequences in \([0, 1]\) satisfying the conditions (iv)–(vi) and \( \sum_{n=1}^{\infty} (1 - \beta_n^i) < \infty \). For arbitrary \( x_0 \in K \), define the sequence \( \{ x_n \} \) by (4.1). Then \( \{ x_n \} \) converges strongly to the common fixed point of \( \bigcap_{i=1}^{p} F(T_i) \cap F(T) \neq \emptyset \).

**Proof.** By applying Theorem 3.2 under assumption that \( T \) and \( T_1 \) are continuous strictly hemicontractive mappings, we obtain Theorem 4.5 which proves strong convergence of the iteration process defined by (4.1). Consider by taking \( T_1 = T \) and \( v_n = y_n^1 \),

\[
\| v_n - x_n \| = \| y_n^1 - x_n \| \\
= \| \beta_n^1 x_{n-1} + \left( 1 - \beta_n^1 \right) T_2 y_n^2 - x_n \| \\
= \| \beta_n^1 x_{n-1} - x_n \| + \left( 1 - \beta_n^1 \right) \| T_2 y_n^2 - x_n \| \\
\leq \beta_n^1 \| x_{n-1} - x_n \| + \left( 1 - \beta_n^1 \right) \| T_2 y_n^2 - x_n \| \\
\leq \beta_n^1 \| x_{n-1} - x_n \| + M'(1 - \beta_n^1).
\]

From (4.5) and the condition \( \sum_{n=1}^{\infty} (1 - \beta_n^1) < \infty \), we obtain

\[
\sum_{n=1}^{\infty} \| v_n - x_n \| < \infty.
\]

This completes the proof. \( \square \)
Corollary 4.6. Let \( X \) be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let \( K \) be a nonempty closed bounded convex subset of \( X \) and let \( T, T_1, T_2, \ldots, T_p : K \to K \) be \( p + 1 \) mappings. Let \( T, T_1 \) be Lipschitz strictly hemicontractive mappings. Let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\alpha_n\} \), \( \{\beta_n\}, \{\gamma_n\} \) be real sequences in \([0, 1]\) satisfying the conditions (iv)-(vi) and \( \sum_{n=1}^{\infty} (1 - \gamma_n) < \infty \). For arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) by (4.1). Then \( \{x_n\} \) converges strongly to the common fixed point of \( \bigcap_{i=1}^{p} F(T_i) \cap F(T) \neq \emptyset \).

References
