Research Article

Sufficient and Necessary Conditions of Complete Convergence for Weighted Sums of PNQD Random Variables

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The complete convergence for pairwise negative quadrant dependent (PNQD) random variables is studied. So far there has not been the general moment inequality for PNQD sequence, and therefore the study of the limit theory for PNQD sequence is very difficult and challenging. We establish a collection that contains relationship to overcome the difficulties that there is no general moment inequality. Sufficient and necessary conditions of complete convergence for weighted sums of PNQD random variables are obtained. Our results generalize and improve those on complete convergence theorems previously obtained by Baum and Katz (1965) and Wu (2002).

1. Introduction and Lemmas

Random variables $X$ and $Y$ are said to be negative quadrant dependent (NQD) if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y),$$  \hspace{1cm} (1.1)

for all $x, y \in \mathbb{R}$. A sequence of random variables $\{X_n; n \geq 1\}$ is said to be pairwise negative quadrant dependent (PNQD) if every pair of random variables in the sequence is NQD. This definition was introduced by Lehmann [1]. Obviously, PNQD sequence includes many negatively associated sequences, and pairwise independent random sequence is the most special case.

In many mathematics and mechanic models, a PNQD assumption among the random variables in the models is more reasonable than an independence assumption. PNQD series have received more and more attention recently because of their wide applications.
Lemma 1.2. Let \( X \) and \( Y \) be NQD random variables. Then

(i) \( \text{cov}(X, Y) \leq 0 \),
(ii) \( P(X > x, Y > y) \leq P(X > x)P(Y > y) \), for all \( x, y \in R \),
(iii) if \( f \) and \( g \) are Borel functions, both of which being monotone increasing (or both are monotone decreasing), then \( f(X) \) and \( g(Y) \) are NQD.

Lemma 1.2 (see [6, Lemma 2]). Let \( \{X_n; n \geq 1\} \) be a sequence of PNQD random variables with \( EX_n = 0 \), \( EX_n^2 < \infty \), \( T_j(k) = \sum_{i=j+1}^{j+k} X_i \), \( j \geq 0 \). Then

\[
E(T_j(k))^2 \leq \sum_{i=j+1}^{j+k} EX_i^2,
\]
\[
E \max_{1 \leq k \leq n} (T_j(k))^2 \leq \frac{4 \log^2 n}{\log 2} \sum_{i=j+1}^{j+n} EX_i^2.
\]  

Lemma 1.3 (see [2, Lemma 1]). (i) If \( \sum_{n=1}^{\infty} P(A_n) < \infty \), then \( P(A_n, i.o.) = 0 \).
(ii) if \( P(A_k A_m) \leq P(A_k)P(A_m), \ k \neq m, \) and \( \sum_{n=1}^{\infty} P(A_n) = \infty \), then \( P(A_n, i.o.) = 1 \).

Lemma 1.4. Let \( \{X_n; n \geq 1\} \) be a sequence of PNQD random variables. Then for any \( x \geq 0 \), there exists a positive constant \( c \) such that for all \( n \geq 1 \),

\[
\left(1 - P\left(\max_{1 \leq k \leq n} |X_k| > x\right)\right)^2 \sum_{k=1}^{n} P(|X_k| > x) \leq c P\left(\max_{1 \leq k \leq n} |X_k| > x\right). \tag{1.3}
\]

Proof. We can prove the Lemma by Lemma A.6 of Zhang and Wen [11].
2. Main Results and the Proof

In the following, the symbol $c$ stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ ($a_n \gg b_n$) denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ ($a_n \geq cb_n$) for all sufficiently large $n$, and let $X_i < X$ ($X_i > X$) denote that there exists a constant $c > 0$ such that $P(|X_i| > x) \leq cP(|X| > x)$ ($P(|X_i| > x) \geq cP(|X| > x)$) for all $i \geq 1$ and $x > 0$.

**Theorem 2.1.** Let $\{X_n; n \geq 1\}$ be a sequence of PNQD random variables with $X_i < X$. Let $\{a_{nk}; k \leq n, n \geq 1\}$ be a sequence of real numbers such that

$$|a_{nk}| \ll n^{-\alpha}, \quad \forall k \leq n, \quad n \geq 1. \quad (2.1)$$

Let for $\alpha p > 1$, $0 < p < 2$, $\alpha > 0$, and $EX_i = 0$, for $\alpha \leq 1$. If

$$E|X|^p < \infty, \quad (2.2)$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} |S_{nk}| > \epsilon \right) < \infty, \quad \forall \epsilon > 0, \quad (2.3)$$

where $S_{nk} = \sum_{i=1}^{k} a_{ni} X_i$.

**Theorem 2.2.** Let $\{X_n; n \geq 1\}$ be a sequence of PNQD random variables with $X_i > X$. Let $\{a_{nk}; k \leq n, n \geq 1\}$ be a sequence of real numbers such that $|a_{nk}| \gg n^{-\alpha}$, for all $k \leq n, n \geq 1$. Let for $\alpha > 0$, $\alpha p > 1$, $0 < p < 2$. If (2.3) holds, then (2.2) holds.

**Remark 2.3.** Taking $a_{ni} = n^{-\alpha}$, for all $i \leq n, n \geq 1$ in Theorem 2.1, then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} |S_{nk}| > \epsilon \right) = \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} n^{-\alpha} \left| \sum_{i=1}^{k} X_i \right| > \epsilon \right)$$

$$= \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right| > \epsilon n^{-\alpha} \right). \quad (2.4)$$

Hence, Theorem 4 in Wu [6] is a particular case of our Theorem 2.1.

**Remark 2.4.** When $\{X_n; n \geq 1\}$ is i.i.d. and $a_{ni} = n^{-\alpha}$, for all $i \leq n, n \geq 1$, then Theorems 2.1 and 2.2 become Baum and Katz [10] complete convergence theorem. Hence, our Theorems 2.1 and 2.2 improve and extend the well-known Baum and Katz theorem.
Proof of Theorem 2.1. Without loss of generality, assume that $a_{nk} > 0$ for $k \leq n$, $n \geq 1$. Let $q > 0$ such that $(1 + (1/ap))/2 < q < 1$. For all $i \leq n$, let

$$Y_{ni} = -a_{ni}^{-1}n^{\alpha(q-1)}I(a_{ni}X_i < -n^{\alpha(q-1)}) + X_i I(a_{ni}|X_i| \leq n^{\alpha(q-1)})$$

$$+ a_{ni}^{-1}n^{\alpha(q-1)}I(a_{ni}X_i > n^{\alpha(q-1)}),$$

(2.5)

$$U_{nk} = \sum_{j=1}^{k} a_{ni}Y_{ni}.$$  

Write

$$A_n = \bigcup_{j=1}^{n} \{|a_{nj}X_j| \geq \varepsilon\},$$

$$B_n = \bigcup_{1 \leq i < j \leq n} \left(\left(a_{ni}X_i > n^{\alpha(q-1)}, a_{nj}X_j > n^{\alpha(q-1)}\right) \cup \left(a_{ni}X_i < -n^{\alpha(q-1)}, a_{nj}X_j < -n^{\alpha(q-1)}\right)\right).$$

(2.6)

Firstly, we prove that

$$\left(\max_{1 \leq k \leq n} |S_{nk}| < 6\varepsilon\right) \supseteq A_n^c \cap \left(\max_{1 \leq k \leq n} |U_{nk}| < 2\varepsilon\right) \cap B_n^c$$

$$= \bigcap_{j=1}^{n} \{|a_{nj}X_j| < \varepsilon\} \cap \left(\max_{1 \leq k \leq n} |U_{nk}| < 2\varepsilon\right) \cap$$

$$\bigcap_{1 \leq i < j \leq n} \left[\left(a_{ni}X_i \leq n^{\alpha(q-1)}\right) \cup \left(a_{nj}X_j \leq n^{\alpha(q-1)}\right)\right]$$

$$\cap \left(\left(a_{ni}X_i \geq -n^{\alpha(q-1)}\right) \cup \left(a_{nj}X_j \geq -n^{\alpha(q-1)}\right)\right)$$

$$\equiv D_n.$$  

(2.7)

For any $\omega \in D_n$, we have

$$|a_{nj}X_j| < \varepsilon, \quad |a_{nj}Y_{nj}| \leq |a_{nj}X_j| < \varepsilon, \quad \forall 1 \leq j \leq n, \quad \max_{1 \leq k \leq n} |U_{nk}| < 2\varepsilon,$$  

(2.8)

and for any $1 \leq i < j \leq n$,

$$a_{ni}X_i \leq n^{\alpha(q-1)}, \quad \text{or} \quad a_{nj}X_j \leq n^{\alpha(q-1)},$$

$$a_{ni}X_i \geq -n^{\alpha(q-1)}, \quad \text{or} \quad a_{nj}X_j \geq -n^{\alpha(q-1)}.$$  

(2.9)
Hence

\[ a = \| \{ i : 1 \leq i \leq n, a_{n_i} X_i(\omega) > n^{a(q-1)} \} \| \leq 1, \]

\[ b = \| \{ i : 1 \leq i \leq n, a_{n_i} X_i(\omega) < -n^{a(q-1)} \} \| \leq 1, \]

(2.10)

where the symbol \( \| A \| \) denotes the number of elements in the set \( A \).

When \( a = b = 0 \), then \( |a_{n_i} X_i(\omega)| \leq n^{a(q-1)} \) for any \( 1 \leq i \leq n \); thus, \( Y_m(\omega) = X_i(\omega) \), and therefore by (2.8),

\[ \max_{1 \leq k \leq n} |S_{nk}| = \max_{1 \leq k \leq n} |U_{nk}| < 2\varepsilon < 6\varepsilon. \]  

(2.11)

When \( a = 1, b = 0 \) (or \( a = 0, b = 1 \)), then there exists only one \( i_0 : 1 \leq i_0 \leq n \) such that \( a_{n_{i_0}} X_{i_0}(\omega) > n^{a(q-1)} \) (or \( a_{n_{i_0}} X_{i_0}(\omega) < -n^{a(q-1)} \)), the remaining \( j \), \( |a_{n_j} X_j(\omega)| \leq n^{a(q-1)} \); thus, \( X_j(\omega) = Y_{n_j}(\omega) \). If \( 1 \leq k \leq i_0 - 1 \), then \( S_{nk}(\omega) = U_{nk}(\omega) \). If \( i_0 \leq k \leq n \), then by (2.8),

\[ \max_{1 \leq k \leq n} |S_{nk}(\omega)| = \max_{1 \leq k \leq n} \left| \sum_{1 \leq i \leq n, i \neq i_0} a_{n_i} X_i(\omega) + a_{n_{i_0}} X_{i_0}(\omega) \right| \]

\[ = \max_{1 \leq k \leq n} \left| \sum_{1 \leq i \leq n} a_{n_i} Y_{n_i}(\omega) - a_{n_{i_0}} Y_{n_{i_0}}(\omega) + a_{n_{i_0}} X_{i_0}(\omega) \right| \]

\[ \leq \max_{1 \leq k \leq n} \left| \sum_{1 \leq i \leq n} a_{n_i} Y_{n_i}(\omega) \right| + |a_{n_{i_0}} Y_{n_{i_0}}(\omega)| + |a_{n_{i_0}} X_{i_0}(\omega)| \]

\[ < 2\varepsilon + \varepsilon + \varepsilon < 6\varepsilon. \]  

(2.12)

When \( a = b = 1 \), then there exist \( 1 \leq i_1, i_2 \leq n \) such that \( a_{n_{i_1}} X_{i_1}(\omega) > n^{a(q-1)} \), \( a_{n_{i_2}} X_{i_2}(\omega) < -n^{a(q-1)} \), the remaining \( j \), \( |a_{n_j} X_j(\omega)| \leq n^{a(q-1)} \); thus, \( X_j(\omega) = Y_{n_j}(\omega) \). Without loss of generality, assume that \( i_1 \leq i_2 \). If \( 1 \leq k \leq i_1 - 1 \), then \( S_{nk}(\omega) = U_{nk}(\omega) \); if \( i_1 \leq k < i_2 \), then by (2.8),

\[ \max_{1 \leq k \leq n} |S_{nk}(\omega)| \leq \max_{1 \leq k \leq n} |U_{nk}(\omega)| + |a_{n_{i_1}} Y_{n_{i_1}}(\omega)| + |a_{n_{i_1}} X_{i_1}(\omega)| \]

\[ < 2\varepsilon + \varepsilon + \varepsilon < 6\varepsilon. \]  

(2.13)

If \( k \geq i_2 \), then by (2.8),

\[ \max_{1 \leq k \leq n} |S_{nk}(\omega)| = \max_{1 \leq k \leq n} \left| \sum_{1 \leq i \leq n, i \neq i_1, i_2} a_{n_i} X_i(\omega) + a_{n_{i_1}} X_{i_1}(\omega) + a_{n_{i_2}} X_{i_2}(\omega) \right| \]

\[ \leq \max_{1 \leq k \leq n} |U_{nk}(\omega)| + |a_{n_{i_1}} Y_{n_{i_1}}(\omega)| + |a_{n_{i_2}} Y_{n_{i_2}}(\omega)| \]
+ |a_{n_i}X_i(\omega)| + |a_{n_i}X_i(\omega)| < 6\varepsilon. \tag{2.14}

Hence, (2.7) holds, that is:

\[ \left( \max_{1 \leq k \leq n} |S_{nk}| \geq 6\varepsilon \right) \subseteq A_n \bigcup \left( \max_{1 \leq k \leq n} |U_{nk}| \geq 2\varepsilon \right) \bigcup B_n. \tag{2.15} \]

Therefore, in order to prove (2.3), we only need to prove that

\[ \sum_{n=1}^{\infty} n^{ap-2} P(A_n) < \infty, \tag{2.16} \]

\[ \sum_{n=1}^{\infty} n^{ap-2} P(B_n) < \infty, \tag{2.17} \]

\[ \sum_{n=1}^{\infty} n^{ap-2} P\left( \max_{1 \leq k \leq n} |U_{nk}| \geq 2\varepsilon \right) < \infty, \quad \forall \varepsilon > 0. \tag{2.18} \]

By (2.1), (2.2), \( X_i < X \), and \( ap > 1 \),

\[ \sum_{n=1}^{\infty} n^{ap-2} P(A_n) \leq \sum_{n=1}^{\infty} n^{ap-2} \sum_{j=1}^{n} P\left( |a_{nj}X_j| \geq \varepsilon \right) \]

\[ \leq \sum_{n=1}^{\infty} n^{ap-2} \sum_{j=1}^{n} P\left( |X_j| \geq \varepsilon a_{nj}^{-1} \geq \varepsilon cn^a \right) \]

\[ \ll \sum_{n=1}^{\infty} n^{ap-1} P(|X| \geq \varepsilon cn^a) \]

\[ = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} P\left( \varepsilon cj^a \leq |X| < \varepsilon c(j+1)^a \right) \]

\[ \leq \sum_{j=1}^{\infty} \sum_{n=1}^{j} \sum_{n=1}^{\infty} n^{ap-1} P\left( \varepsilon cj^a \leq |X| < \varepsilon c(j+1)^a \right) \]

\[ \ll E|X|^p < \infty. \]

That is, (2.16) holds.
By Lemma 1.1(ii), \( X_i < X \), and the definition of \( q \), \( ap(1 - 2q) < -1 \),

\[
\sum_{n=1}^{\infty} n^{ap-2} P(B_n) \leq \sum_{n=1}^{\infty} n^{ap-2} \sum_{1 \leq i < j < n} \left( P\left( a_{ni} X_i > n^{a(q-1)} \right) P\left( a_{nj} X_j > n^{a(q-1)} \right) \right) \\
+ P\left( a_{ni} X_i < -n^{a(q-1)} \right) P\left( a_{nj} X_j < -n^{a(q-1)} \right)
\]

\[
\ll \sum_{n=1}^{\infty} n^{ap} P^2(|X| > cn^q) \leq \sum_{n=1}^{\infty} n^{ap} n^{-2apq} (E|X|^p)^2
\]

\[
\ll \sum_{n=1}^{\infty} n^{ap(1-2q)} < \infty.
\]

That is, (2.17) holds.

In order to prove (2.18), firstly, we prove that

\[
\max_{1 \leq k \leq n} |E\sum_{i=1}^{k} a_{ni} Y_{ni}| \rightarrow 0, \quad n \rightarrow \infty.
\]

(2.21)

(i) When \( \alpha \leq 1 \), then \( p > 1/\alpha \geq 1 \); from \( EX_i = 0 \) and the definition of \( q \), we have \( q < 1 \), \( apq > ap + 1 - apq = 1 + ap(1 - q) > 1 \):

\[
\max_{1 \leq k \leq n} \left| E \sum_{i=1}^{k} a_{ni} Y_{ni} \right| \\
\leq \sum_{i=1}^{n} a_{ni} |EY_{ni}| = \sum_{i=1}^{n} a_{ni} |E(X_i - Y_{ni})| \\
\leq \sum_{i=1}^{n} a_{ni} \left( E|X_i + a_{ni}^{-1} n^{a(q-1)} (I_{(a_{ni} X_i < -n^{a(q-1)})} + E|X_i - a_{ni}^{-1} n^{a(q-1)} |I_{(a_{ni} X_i > n^{a(q-1)})}) \right) \\
\ll \sum_{i=1}^{n} a_{ni} E|X_i| I_{(a_{ni} X_i > n^{a(q-1)})} \leq \sum_{i=1}^{n} a_{ni} E|X_i| \left( \frac{a_{ni}}{n^{a(q-1)}} \right)^{p-1} \\
= \sum_{i=1}^{n} a_{ni} E|X_i|^{p} n^{-a(1-q)(p-1)} \\
\ll n^{-ap+1+ap-a-apq+aq} = n^{-(apq-1)-a(1-q)} \\
\rightarrow 0, \quad n \rightarrow \infty.
\]

(ii) When \( \alpha > 1 \), and \( p \geq 1 \), then \( E|X| < \infty \) from (2.2), thus,

\[
\max_{1 \leq k \leq n} \left| E \sum_{i=1}^{k} a_{ni} Y_{ni} \right| \leq \sum_{i=1}^{n} a_{ni} E|X| \ll n^{-\alpha+1} \rightarrow 0, \quad n \rightarrow \infty.
\]  

(2.23)
(iii) When $\alpha > 1$, and $p < 1$, by $-(ap-1)-\alpha(1-q)(1-p) < 0$, and $-\alpha(1-q)-(apq-1) < 0$, we get

$$
\max_{1 \leq k \leq n} \left| E \sum_{i=1}^{k} a_{ni} Y_{ni} \right| \leq \sum_{i=1}^{n} a_{ni} \left( E|X_i|I(a_{ni}|X_i| \leq n^{p+q(1)}) + a_{ni}^{-1} n^{\alpha(q-1)} P\left(|a_{ni}|X_i| > n^{\alpha(q-1)}\right) \right) 
\leq \sum_{i=1}^{n} a_{ni}^{p} E|X_i|^p I(a_{ni}|X_i| \leq n^{p+q(1)}) + \sum_{i=1}^{n} n^{\alpha(q-1)} a_{ni}^{-1} n^{-apq(q-1)} E|X_i|^p 
\ll n^{-(ap-1)-\alpha(1-p)(1-q)} + n^{-\alpha(1-q)-(apq-1)} \rightarrow 0. 
$$

Hence, (2.21) holds; that is, for any $\varepsilon > 0$, we have $\max_{1 \leq k \leq n}|EU_{nk}| < \varepsilon$ for all sufficiently large $n$. Thus,

$$
P\left( \max_{1 \leq j \leq n} |U_{nj}| \geq 2\varepsilon \right) \leq P\left( \max_{1 \leq j \leq n} |U_{nj} - EU_{nj}| > \varepsilon \right). 
$$

Let $\bar{Y}_{ni} = Y_{ni} - EU_{ni}$. Obviously, $Y_{ni}$ is monotonic on $X_i$. By Lemma 1.1(iii), $\{|Y_{ni}|; n \geq 1, i \leq n\}$ is also a sequence of PNQD random variables with $EU_{ni} = 0$, by Lemma 1.2 and $-1 - \alpha(1-q)(2-p) < -1$:

$$
\sum_{n=1}^{\infty} n^{ap-2} P\left( \max_{1 \leq j \leq n} |U_{nj} - EU_{nj}| > \varepsilon \right) 
\ll \sum_{n=1}^{\infty} n^{ap-2} \log^2 n \sum_{j=1}^{n} E a_{nj}^2 Y_{nj}^2 
\ll \sum_{n=1}^{\infty} n^{ap-2} \log^2 n \sum_{j=1}^{n} \left( E a_{nj}^2 X_{j}^2 I(a_{nj}|X_j| \leq n^{ap(1-q)}) + n^{2\alpha(q-1)} P\left(a_{nj}|X_j| > n^{\alpha(q-1)}\right) \right) 
\leq \sum_{n=1}^{\infty} n^{ap-2} \log^2 n \sum_{j=1}^{n} \left( E a_{nj}^2 X_{j}^2 P\left(a_{nj}|X_j| > n^{\alpha(q-1)}\right) + n^{2\alpha(q-1)-ap(q-1)} E a_{nj}^2 X_{j}^2 \right) 
\ll \sum_{n=1}^{\infty} \left( n^{ap-1-ap+\alpha(q-1)(2-p)} + n^{-1+apq+2\alpha q(2-p) - apq(q-1)} \right) \log^2 n 
= 2 \sum_{n=1}^{\infty} n^{-1-(1-q)(2-p)} \log^2 n 
< \infty. 
$$

This completes the proof of Theorem 2.1. □
Proof of Theorem 2.2. Noting that $\max_{1 \leq k \leq n}|a_{nk} X_k| \leq 2 \max_{1 \leq k \leq n}|S_{nk}|$ and $|a_{nk}| \gg n^{-\alpha}$, from (2.3),

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} |X_k| > \varepsilon n^\alpha\right) < \infty, \quad \forall \varepsilon > 0. \quad (2.27)$$

Thus, by $\alpha p - 2 > -1$, we get

$$\sum_{j=1}^{\infty} P\left(\max_{1 \leq k \leq 2^j} |X_k| > \varepsilon 2^{\alpha(j+1)}\right) \ll \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq k \leq n} |X_k| > \varepsilon n^\alpha\right) < \infty, \quad \forall \varepsilon > 0. \quad (2.28)$$

This implies that

$$\max_{2 \leq m \leq 2^n} P\left(\max_{1 \leq j \leq n} |X_j| > \varepsilon 2^{\alpha(m+1)}\right) \leq P\left(\max_{1 \leq j \leq 2^m} |X_j| > \varepsilon 2^{\alpha(m+1)}\right) \rightarrow 0. \quad (2.29)$$

Hence, for all sufficiently large $n$,

$$P\left(\max_{1 \leq j \leq n} |X_j| > \varepsilon 2^{\alpha n^\alpha}\right) < \frac{1}{2}. \quad (2.30)$$

By Lemma 1.4,

$$\sum_{k=1}^{n} P(|X_k| > \varepsilon n^\alpha) \leq 4cP\left(\max_{1 \leq k \leq n} |X_k| > \varepsilon n^\alpha\right), \quad \forall \varepsilon > 0, \quad (2.31)$$

which together with (2.27),

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|X_k| > \varepsilon n^\alpha) < \infty, \quad \forall \varepsilon > 0. \quad (2.32)$$

By $X_k \succ X$, we obtain

$$E|X|^p \ll \sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| > \varepsilon n^\alpha) < \infty. \quad (2.33)$$

This completes the proof of Theorem 2.2. \qed

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