Research Article

Necessary and Sufficient Condition for Stability of Generalized Expectation Value

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A class of generalized definitions of expectation value is often employed in nonequilibrium statistical mechanics for complex systems. Here, the necessary and sufficient condition is presented for such a class to be stable under small deformations of a given arbitrary probability distribution.

Given a probability distribution \( \{ p_i \}_{i=1,2,...,W} \), that is, \( 0 \leq p_i \leq 1 \ (i = 1, 2, \ldots, W) \) and \( \sum_{i=1}^{W} p_i = 1 \), the ordinary expectation value of a quantity \( Q \) of a system under consideration is defined by

\[
\sum_{i=1}^{W} P_i Q_i,
\]

where \( P_i = \sum_{j=1}^{W} \phi(p_j) \), with a nonnegative function \( \phi \). In the special case when \( \phi(x) = x \), \( \langle Q \rangle_{\phi} \) is reduced to the ordinary expectation value mentioned above.
Consider measurements of a certain quantity of a system to obtain information about the probability distribution. Repeated measurements should be performed on the system, which is identically prepared each time. Suppose that two probability distributions, \( \{ p_i \}_{i=1,2,\ldots, W} \) and \( \{ p'_i \}_{i=1,2,\ldots, W'} \), are obtained through the measurements. They may slightly be different from each other, in general. If such measurements make sense, then the expectation values, \( \langle Q \rangle [p] \) and \( \langle Q \rangle [p'] \), calculated from these two distributions should also be close to each other. This condition, which implies “experimental robustness,” is represented as follows.

**Definition** (stability). An expectation value \( \langle Q \rangle [p] \) is said to be stable, if the following predicate holds for any pair of probability distributions, \( \{ p_i \}_{i=1,2,\ldots, W} \) and \( \{ p'_i \}_{i=1,2,\ldots, W'} \):

\[
\left( \forall \varepsilon > 0 \right) \left( \exists \delta > 0 \right) \left( \forall W \right) \left( \| p - p' \|_1 < \delta \Rightarrow | \langle Q \rangle [p] - \langle Q \rangle [p'] | < \varepsilon \right).
\]

(3)

Here, \( \| p - p' \|_1 = \sum_{i=1}^{W} | p_i - p'_i | \) is the \( l^1 \)-norm describing the distance between these two probability distributions. One might consider norms of other kinds, but what is physically relevant to discrete systems is the present \( l^1 \)-norm [5]. Equation (3) is analogous to Lesche’s stability condition on entropic functionals [5], which has recently been revisited in the literature [6–11] (note that the discussion in [8] is corrected in [9]). This concept of stability is actually equivalent to that of uniform continuity.

In recent papers [12, 13], it has been shown that the generalized expectation value in (1) with a specific class, \( \phi(x) = x^q \) (\( q > 0 \)), (the associated expectation value being termed the \( q \)-expectation value), is not stable unless \( q = 1 \). This result needs the \( q \)-expectation-value formalism of nonextensive statistical mechanics [1, 2] to be reconsidered. In addition, the result is supported by Boltzmann-like kinetic theory in an independent manner [14].

Here, it seems appropriate to make some comments on the latest situation of the problems concerning stabilities of entropic functionals and generalized expectation values. The authors of [15, 16] have presented discussions which aim to rescue the \( q \)-expectation values from the difficulties of their instability pointed out in [12]. Those authors insist that the \( q \)-expectation values can be stable in both the finite-\( W \) and continuous cases. Such possibilities are, however, fully refuted by the work in [13] both physically and mathematically, and the controversy seems to have been terminated with that work. The case of the continuous variables has further been carefully examined in a recent paper [17], where the so-called Tsallis \( q \)-entropies [1, 2] do not have the continuous limit in consistency with the physical principles such as the thermodynamic laws (see also [18, 19]). These controversies have led the researchers to give up the traditional form of nonextensive statistical mechanics based on the \( q \)-entropies and \( q \)-expectation values and to examine other entropic functionals combined with the ordinary definition of expectation values [20] (see also [21, 22]). Thus, it seems that nonextensive statistical mechanics has to be fully reexamined, theoretically.

In this paper, we present the necessary and sufficient condition for \( \langle Q \rangle_{\phi}[p] \) in (1) to be stable.

Our main result is as follows.

**Theorem.** Let \( \phi \) be nonnegative and continuous on \( [0,1] \), differentiable on \( (0,1) \), and satisfy the condition that \( \phi(x) = 0 \Leftrightarrow x = 0 \). And, let \( Q = \{ Q_i \}_{i=1,2,\ldots, W} \) be a random variable. Then, \( \langle Q \rangle_{\phi}[p] \) in (1) is stable, if and only if \( \lim_{x \to 0+} \phi(x)/x \in (0, \infty) \).
Proof. First, assume that \( \lim_{x \to 0} \phi(x)/x = a > 0 \). Then, there exists \( \delta_1 > 0 \) such that

\[
a - \frac{a}{2} < \frac{\phi(x)}{x} < a + \frac{a}{2} \quad (\forall x \in (0, \delta_1]).
\]

(4)

\( \phi(x)/x \) does not vanish because of the condition \( \phi(x) = 0 \iff x = 0 \). Therefore, there exists \( b > 0 \) such that

\[
\frac{\phi(x)}{x} \geq b \quad (\forall x \in (\delta_1, 1]).
\]

(5)

Putting \( c = \min\{a/2, b\} \) we have

\[
cx \leq \phi(x) \quad (\forall x \in [0, 1]).
\]

(6)

Consequently, for an arbitrarily large \( W \) and an arbitrary probability distribution \( \{p_i\}_{i=1,2,\ldots,W} \), we obtain

\[
\frac{1}{\sum_{i=1}^{W} \phi(p_i)} \leq c.
\]

(7)

From the mean value theorem, it follows that

\[
|\phi(p_i) - \phi(p'_i)| \leq |p_i - p'_i| \cdot \sup_{x \in (0,1)} |\phi'(x)|,
\]

where \( \phi'(x) \) is the derivative of \( \phi(x) \) with respect to \( x \). For \( \varepsilon > 0 \), we put

\[
\delta = \inf\left( \delta_1, \frac{ce}{2|Q_{\text{max}}| \cdot \left( \sup_{x \in (0,1)} |\phi'(x)| \right)} \right),
\]

(9)

where \( Q_{\text{max}} = \max \{Q_i\}_{i=1,2,\ldots,W} \). Now, for \( \|p - p'\|_1 < \delta \), we have

\[
\left| \langle Q \rangle_{\phi}[p] - \langle Q \rangle_{\phi}[p'] \right|
\]

\[
= \frac{1}{\sum_{i=1}^{W} \phi(p_i) \cdot \sum_{j=1}^{W} \phi(p'_j)} \left| \sum_{i=1}^{W} Q_i \left\{ \phi(p_i) \sum_{j=1}^{W} \phi(p'_j) - \phi(p'_i) \sum_{j=1}^{W} \phi(p_j) \right\} \right|
\]

\[
\leq \frac{1}{\sum_{i=1}^{W} \phi(p_i) \cdot \sum_{j=1}^{W} \phi(p'_j)}
\]

\[
\times \left[ \sum_{i=1}^{W} |Q_i| \left\{ |\phi(p_i) - \phi(p'_i)| \sum_{j=1}^{W} \phi(p'_j) + \phi(p'_i) \sum_{j=1}^{W} \phi(p_j) - \sum_{j=1}^{W} \phi(p'_i) \right\} \right]
\]
Therefore, \( \langle Q \rangle_\phi[p] \) is stable.

On the other hand, suppose that \( \lim_{x \to 0} \phi(x)/x \notin (0, \infty) \). That is, (i) \( \lim_{x \to 0} \phi(x)/x = 0 \) or (ii) \( \lim_{x \to 0} \phi(x)/x = \infty \). Below, we will examine these cases separately.

(i) Consider the following deformation:

\[
p_i = \frac{1}{W-1}(1 - \delta_{ii}), \quad p_i' = \left(1 - \frac{\delta}{2}\right)p_i + \frac{\delta}{2}\delta_{ii},
\]

which are normalized and satisfy \( \|p - p'\|_1 = \delta \). We have

\[
\sum_{i=1}^{W} \phi(p_i) = (W-1)\phi\left(\frac{1}{W-1}\right), \quad \sum_{i=1}^{W} \phi(p_i') = \phi\left(\frac{\delta}{2}\right) + (W-1)\phi\left(\frac{1}{W-1}\left(1 - \frac{\delta}{2}\right)\right).
\]

Difference of the expectation values is calculated as follows:

\[
\langle Q \rangle_\phi[p] - \langle Q \rangle_\phi[p']
\]

\[
= -\frac{Q_1\phi(\delta/2)}{\phi(\delta/2) + (W-1)\phi((1/(W-1))(1 - \delta/2))} + \left(\sum_{i=2}^{W} Q_i\right)\left\{\frac{1}{W-1} - \frac{\phi((1/(W-1))(1 - \delta/2))}{\phi(\delta/2) + (W-1)\phi((1/(W-1))(1 - \delta/2))}\right\}
\]

\( \varepsilon. \)
\[
\frac{W}{W-1} \left( \overline{Q} - Q_1 \right) \\
\times \frac{\phi(\delta/2)/(1-\delta/2)}{\phi(\delta/2)/(1-\delta/2) + \phi((1/(W-1))(1-\delta/2))/[(1/(W-1))(1-\delta/2)]}
\xrightarrow{W \to \infty} \overline{Q} - Q_1,
\]

(13)

since \( \lim_{x \to +0} \phi(x)/x = 0 \), where \( \overline{Q} \) is the arithmetic mean, \( \overline{Q} = \sum_{i=1}^{W} Q_i/W \). Therefore, \( \langle Q \rangle_{\phi}[p] \) is not stable.

(ii) Consider the following deformation:

\[
p_i = \delta_{i1}, \\
p'_i = \left(1 - \frac{\delta}{2} \frac{W}{W-1}\right)p_i + \frac{\delta}{2} \frac{1}{W-1},
\]

which are also normalized and satisfy \( \|p - p'\|_1 = \delta \). We have

\[
\sum_{i=1}^{W} \phi(p_i) = \phi(1), \\
\sum_{i=1}^{W} \phi(p'_i) = \phi\left(1 - \frac{\delta}{2}\right) + (W-1)\phi\left(\frac{\delta}{2} \frac{1}{W-1}\right).
\]

Difference of the expectation values is calculated as follows:

\[
\langle Q \rangle_{\phi}[p] - \langle Q \rangle_{\phi}[p'] = Q_1 \left\{ 1 - \frac{\phi(1-\delta/2)}{\phi(1-\delta/2) + (W-1)\phi((\delta/2)(1/(W-1)))} \right\} \\
- \left( \sum_{i=2}^{W} Q_i \right) \frac{\phi((\delta/2)(1/(W-1)))}{\phi(1-\delta/2) + (W-1)\phi((\delta/2)(1/(W-1)))}
\]

\[
= \frac{W}{W-1} \left( Q_1 - \overline{Q} \right) \\
\times \frac{\phi((\delta/2)(1/(W-1))) / [(\delta/2)(1/(W-1))] }{\phi(1-\delta/2)/(\delta/2) + \phi((\delta/2)(1/(W-1))) / [(\delta/2)(1/(W-1))]}
\xrightarrow{W \to \infty} Q_1 - \overline{Q},
\]

(16)

since \( \lim_{x \to +0} \phi(x)/x = \infty \). Therefore, \( \langle Q \rangle_{\phi}[p] \) is not stable.

\( \square \)

In the above proof, we have employed the specific deformations of the probability distributions as the counterexamples, which are considered in [5]. It is pointed out in [13] that these deformed distributions may experimentally be generated.
Finally, we mention a couple of simple stable examples.

**Example 1.**

\[
\phi(x) = e^x - 1.
\]

(17)

**Example 2.**

\[
\phi(x) = \ln(1 + x^\alpha),
\]

(18)

which yields a stable generalized expectation value, if and only if \(\alpha = 1\).

On the other hand, as mentioned earlier, the \(q\)-expectation value is not stable, since \(\phi(x) = x^q (q > 0, q \neq 1)\) does not satisfy the condition \(\lim_{x \to 0} \phi(x) / x \in (0, \infty)\).

In conclusion, we have considered a class of generalized definitions of expectation value that are often employed in nonequilibrium statistical mechanics for complex systems, and have presented the necessary and sufficient condition for such a class to be stable under small deformations of a given arbitrary probability distribution.

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**References**


