Research Article

Starlikeness Properties of a New Integral Operator for Meromorphic Functions

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Received 3 March 2011; Revised 15 April 2011; Accepted 3 June 2011

Academic Editor: Pablo González-Vera

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We define here an integral operator $\mathcal{K}_{\gamma_1,\ldots,\gamma_n}$ for meromorphic functions in the punctured open unit disk. Several starlikeness conditions for the integral operator $\mathcal{K}_{\gamma_1,\ldots,\gamma_n}$ are derived.

1. Introduction

Let $\Sigma$ denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We say that a function $f \in \Sigma$ is meromorphic starlike of order $\alpha$ ($0 \leq \alpha < 1$), and belongs to the class $\Sigma^*(\alpha)$, if it satisfies the inequality

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$
A function $f \in \Sigma$ is a meromorphic convex function of order $\alpha$ ($0 \leq \alpha < 1$), if $f$ satisfies the following inequality

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,$$  \hspace{1cm} (1.4)

and we denote this class by $\Sigma_k(\alpha)$.

Analogous to the integral operator defined by Breaz et al. [1] on the normalized analytic functions, we now define the following integral operator on the space meromorphic functions in the class $\Sigma$.

**Definition 1.1.** Let $n \in \mathbb{N}$, $\gamma_i > 0$, $i \in \{1, 2, 3, \ldots, n\}$. We define the integral operator $H_{\gamma_1, \ldots, \gamma_n}(f_1, f_2, \ldots, f_n) : \Sigma^n \to \Sigma$ by

$$H_{\gamma_1, \ldots, \gamma_n}(f_1, \ldots, f_n)(z) = \frac{1}{z^2} \int_{\gamma_1}^{z} \left(-u^2 f_1'(u)\right) \cdots \left(-u^2 f_n'(u)\right)^{\gamma_n} du.$$  \hspace{1cm} (1.5)

For the sake of simplicity, from now on we will write $H_{\gamma_1, \ldots, \gamma_n}(z)$ instead of $H_{\gamma_1, \ldots, \gamma_n}(f_1, \ldots, f_n)(z)$.

By $\Sigma_k(\beta)$ ($-1 \leq \beta < 1$), we denote the class of functions $f \in \Sigma$ such that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| < -\Re\left(\frac{zf''(z)}{f'(z) + \beta} + 1\right).$$  \hspace{1cm} (1.6)

In order to derive our main results, we have to recall here the following preliminary results.

**Lemma 1.2** (see [2]). Suppose that the function $\Psi : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:

$$\Re\{\Psi(is, t)\} \leq 0, \quad (s, t \in \mathbb{R}; \quad t \leq \frac{1 + s^2}{2}).$$  \hspace{1cm} (1.7)

If the function $p(z) = 1 + p_1z + \cdots$ is analytic in $\mathbb{U}$ and

$$\Re\{\Psi(p(z), zp'(z))\} > 0, \quad (z \in \mathbb{U}),$$

then,

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$  \hspace{1cm} (1.9)

**Proposition 1.3** (see [3]). If $f \in \Sigma$ satisfying

$$-\Re\left\{\frac{zf''(z) + 3f'(z)}{zf'(z) + 2f(z)}\right\} > \alpha, \quad 0 \leq \alpha < 1,$$

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < 1,$$  \hspace{1cm} (1.10)
then,
\[-\Re\left\{ z \frac{f'(z)}{f(z)} \right\} > \alpha. \quad (1.11)\]

2. Starlikeness of the Operator \( \mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) \)

In this section, we investigate sufficient conditions for the integral operator \( \mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) \) which is defined in Definition 1.1, to be in the class \( \Sigma^*(\alpha), \ 0 \leq \alpha < 1. \)

**Theorem 2.1.** Let \( f_i \in \sum, \ \gamma_i > 0 \) for all \( i \in \{1, \ldots, n\}. \) If
\[-\Re\left( z \frac{f''(z)}{f'(z)} \right) > \frac{-1}{n \gamma_i} + 2, \quad (2.1)\]

then \( \mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) \) belongs to \( \Sigma^*(0). \)

**Proof.** On successive differentiation of \( \mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) \), which is defined in (1.5), we get
\[
2z\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) + z^2 \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) = \left( -z^2 f'_1(z) \right)^n \cdots \left( -z^2 f'_n(z) \right)^n,
\]
\[
z^2 \mathcal{H}''_{\gamma_1, \ldots, \gamma_n}(z) + 4z \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)
\]
\[
= \sum_{i=1}^{n} \gamma_i \left( -z^2 f''_1(z) \right)^{n-1} \left( -z^2 f''_i(z) - 2z f'_i(z) \right) \prod_{j=1, j \neq i}^{n} \left( -z^2 f''_j(z) \right)^{\gamma_j}. \quad (2.2)
\]

Then from (2.2), we obtain
\[
\frac{z^2 \mathcal{H}''_{\gamma_1, \ldots, \gamma_n}(z) + 4z \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)}{z^2 \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2z \mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)} = \sum_{i=1}^{n} \gamma_i \left( \frac{f''_i(z)}{f'_i(z)} + 2 \right). \quad (2.3)
\]

By multiplying (2.3) with \( z \) yield,
\[
\frac{z^2 \mathcal{H}''_{\gamma_1, \ldots, \gamma_n}(z) + 4z \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)}{z \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)} = \sum_{i=1}^{n} \gamma_i \left( \frac{zf''_i(z)}{f'_i(z)} + 2 \right). \quad (2.4)
\]

That is equivalent to
\[
\left\{ \frac{z \left( z \mathcal{H}''_{\gamma_1, \ldots, \gamma_n}(z) + 3\mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) \right)}{z \mathcal{H}'_{\gamma_1, \ldots, \gamma_n}(z) + 2\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z)} \right\} + 1 = \sum_{i=1}^{n} \gamma_i \left( \frac{zf''_i(z)}{f'_i(z)} + 2 \right). \quad (2.5)
\]
Or

\[- \left\{ \frac{z (z \mathcal{R}_{1,\ldots,n}(z) + 3 \mathcal{R}_{1,\ldots,n}(z))}{z \mathcal{R}_{1,\ldots,n}(z) + 2 \mathcal{R}_{1,\ldots,n}(z)} \right\} = \sum_{i=1}^{n} y_i \left( -\frac{z f_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.6} \]

We can write the left-hand side of (2.6), as the following:

\[- \left( \frac{z \mathcal{R}_{1,\ldots,n}(z) / \mathcal{R}_{1,\ldots,n}(z)}{(z \mathcal{R}_{1,\ldots,n}(z) / \mathcal{R}_{1,\ldots,n}(z)) + 2} \right)^{\prime} = \sum_{i=1}^{n} y_i \left( -\frac{z f_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.7} \]

We define the regular function \( p \) in \( U \) by

\[ p(z) = -\frac{z \mathcal{R}_{1,\ldots,n}(z)}{\mathcal{R}_{1,\ldots,n}(z)}. \tag{2.8} \]

and \( p(0) = 1 \). Differentiating \( p(z) \) logarithmically, we obtain

\[-p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z \mathcal{R}_{1,\ldots,n}(z)}{\mathcal{R}_{1,\ldots,n}(z)}. \tag{2.9} \]

From (2.7), (2.8), and (2.9), we obtain

\[ p(z) + \frac{zp'(z)}{-p(z) + 2} = \sum_{i=1}^{n} y_i \left( -\frac{z f_i''(z)}{f_i'(z)} \right) - 2 \sum_{i=1}^{n} y_i + 1. \tag{2.10} \]

Let us put

\[ \Psi(u, v) = u + \frac{v}{-u + 2}. \tag{2.11} \]

From (2.1), (2.10), and (2.11), we obtain

\[ \Re\{\Psi(p(z), zp'(z))\} = y_1 \left( -\Re \frac{z f_1''(z)}{f_1'(z)} \right) + \cdots + \left( -\Re \frac{z f_n''(z)}{f_n'(z)} \right) - 2 (y_1 + \cdots + y_n) + 1 \]

\[ > y_1 \left( \frac{-1}{ny_1} + 2 \right) + \cdots + y_n \left( \frac{-1}{ny_n} + 2 \right) - 2 (y_1 + \cdots + y_n) + 1 = 0. \tag{2.12} \]
Now, we proceed to show that

\[ \Re\{\Psi(is,t)\} \leq 0, \quad \left( s, t \in \mathbb{R}; t \leq \frac{-(1 + s^2)}{2} \right). \tag{2.13} \]

Indeed, from (2.11), we have

\[ \Re\{\Psi(is,t)\} = \Re\left\{ is + \frac{t}{-is + 2} \right\} = \frac{2t}{4 + s^2} \leq -\frac{1 + s^2}{4 + s^2} < 0. \tag{2.14} \]

Thus, from (2.12), (2.14), and by using Lemma 1.2, we conclude that \( \Re\{p(z)\} > 0 \), and so

\[ -\Re\left\{ \frac{z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,...,\gamma_n}(z)} \right\} > 0 \tag{2.15} \]

that is, \( \mathcal{H}_{\gamma_1,...,\gamma_n}(z) \) is starlike of order 0.

**Theorem 2.2.** For \( i \in \{1, \ldots, n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Sigma_k(\alpha_i) \) (\( 0 \leq \alpha_i < 1 \)). If \( 0 < \sum_{i=1}^{n} \gamma_i (1 - \alpha_i) \leq 1 \), \( \mathcal{H}_{\gamma_1,...,\gamma_n}(z) \) be the integral operator given by (1.5) and

\[ \left| \frac{z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,...,\gamma_n}(z)} + 1 \right| < 1. \tag{2.16} \]

Then \( \mathcal{H}_{\gamma_1,...,\gamma_n}(z) \) belong to \( \Sigma^{\ast}(\mu) \), where \( \mu = 1 - \sum_{i=1}^{n} \gamma_i (1 - \alpha_i) \).

**Proof.** Following the same steps as in Theorem 2.1, we obtain

\[ -\left\{ \frac{z\left( z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) + 3\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) \right)}{z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,...,\gamma_n}(z)} \right\} = \sum_{i=1}^{n} \gamma_i \left\{ -\left( \frac{zf_i''(z)}{f_i'(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \tag{2.17} \]

Taking the real part of both terms of the last expression, we have

\[ -\Re\left\{ \frac{z\left( z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) + 3\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) \right)}{z\mathcal{H}'_{\gamma_1,...,\gamma_n}(z) + 2\mathcal{H}_{\gamma_1,...,\gamma_n}(z)} \right\} = \sum_{i=1}^{n} \gamma_i \left\{ -\Re\left( \frac{zf_i''(z)}{f_i'(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \tag{2.18} \]
Since $f_i \in \Sigma_k(\alpha_i)$, for $i \in \{1, \ldots, n\}$, we receive

$$-\Re\left\{\frac{z(\mathcal{H}_{y_1, \ldots, y_n}''(z) + 3\mathcal{H}_{y_1, \ldots, y_n}'(z))}{z\mathcal{H}_{y_1, \ldots, y_n}'(z) + 2\mathcal{H}_{y_1, \ldots, y_n}(z)}\right\} > \sum_{i=1}^{n} \gamma_i \alpha_i + 1 - \sum_{i=1}^{n} \gamma_i.$$  \hfill (2.19)

Therefore,

$$-\Re\left\{\frac{z(\mathcal{H}_{y_1, \ldots, y_n}''(z) + 3\mathcal{H}_{y_1, \ldots, y_n}'(z))}{z\mathcal{H}_{y_1, \ldots, y_n}'(z) + 2\mathcal{H}_{y_1, \ldots, y_n}(z)}\right\} > 1 - \sum_{i=1}^{n} \gamma_i (1 - \alpha_i).$$  \hfill (2.20)

Using (2.16), (2.20), and applying Proposition 1.3, we get $\mathcal{H}_{y_1, \ldots, y_n}(z) \in \Sigma^*(\mu)$, where $\mu = 1 - \sum_{i=1}^{n} \gamma_i (1 - \alpha_i)$.

Letting $\alpha_i = \alpha$, $i \in \{1, \ldots, n\}$ in Theorem 2.2, we get the following.

**Corollary 2.3.** For $i \in \{1, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma_k(\alpha)$ ($0 \leq \alpha < 1$). If

$$0 < \sum_{i=1}^{n} \gamma_i \leq \frac{1}{1 - \alpha},$$  \hfill (2.21)

$\mathcal{H}_{y_1, \ldots, y_n}$ be the integral operator given by (1.5) and

$$\left|\frac{\mathcal{H}_{y_1, \ldots, y_n}'(z)}{\mathcal{H}_{y_1, \ldots, y_n}(z)} + 1\right| < 1.$$  \hfill (2.22)

Then $\mathcal{H}_{y_1, \ldots, y_n}(z)$ is starlike of order $1 - (1 - \alpha) \sum_{i=1}^{n} \gamma_i$.

**Theorem 2.4.** For $i \in \{1, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma_k(\beta_i)$ ($-1 \leq \beta_i < 1$). If

$$0 < \sum_{i=1}^{n} \gamma_i (1 - \beta_i) \leq 1,$$  \hfill (2.23)

$\mathcal{H}_{y_1, \ldots, y_n}(z)$ be the integral operator given by (1.5) and

$$\left|\frac{\mathcal{H}_{y_1, \ldots, y_n}'(z)}{\mathcal{H}_{y_1, \ldots, y_n}(z)} + 1\right| < 1.$$  \hfill (2.24)

Then $\mathcal{H}_{y_1, \ldots, y_n}(z)$ is starlike of order $1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i)$. 
Corollary 2.5. Following the same steps as in Theorem 2.1, we obtain

\[
\begin{aligned}
- \left\{ \frac{z \mathcal{K}_{n}''(z) + 3 \mathcal{K}''(z)}{z \mathcal{K}_{n}'(z) + 2 \mathcal{K}'(z)} \right\} &= - \sum_{i=1}^{n} \left( \frac{zf''_i(z) + 2}{f'_i(z)} + 1 \right) \\
&= \sum_{i=1}^{n} \left( - \left( \frac{zf''_i(z)}{f'_i(z)} + \beta_i \right) - 1 \right) + 1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i).
\end{aligned}
\] (2.25)

We calculate the real part from both terms of the above equality and obtain

\[
\begin{aligned}
- \Re \left\{ \frac{z \mathcal{K}_{n}''(z) + 3 \mathcal{K}''(z)}{z \mathcal{K}_{n}'(z) + 2 \mathcal{K}'(z)} \right\} &= \sum_{i=1}^{n} \left( - \Re \left( \frac{zf''_i(z)}{f'_i(z)} + \beta_i \right) - 1 \right) + 1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i).
\end{aligned}
\] (2.26)

Since \( f_i \in \Sigma_{\kappa_i} (\beta_i) \) for all \( i \in \{1, \ldots, n\} \), the above relation then yields

\[
\begin{aligned}
- \Re \left\{ \frac{z \mathcal{K}_{n}''(z) + 3 \mathcal{K}''(z)}{z \mathcal{K}_{n}'(z) + 2 \mathcal{K}'(z)} \right\} &> \sum_{i=1}^{n} \left| \frac{zf''_i(z)}{f'_i(z)} + 2 \right| + 1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i).
\end{aligned}
\] (2.27)

Because \( \sum_{i=1}^{n} |zf''_i(z) / f'_i(z) + 2| \geq 0 \), we obtain that

\[
\begin{aligned}
- \Re \left\{ \frac{z \mathcal{K}_{n}''(z) + 3 \mathcal{K}''(z)}{z \mathcal{K}_{n}'(z) + 2 \mathcal{K}'(z)} \right\} &> 1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i).
\end{aligned}
\] (2.28)

Using (2.24), (2.28) and applying Proposition 1.3, we get \( \mathcal{K}_{n}'(z) \) is a starlike function of order \( 1 - \sum_{i=1}^{n} \gamma_i (1 - \beta_i) \). \( \Box \)

Letting \( \beta_i = \beta \), \( i \in \{1, \ldots, n\} \) in Theorem 2.4, we get the following.

**Corollary 2.5.** For \( i \in \{1, \ldots, n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Sigma_{\kappa_i} (\beta) \) \( (-1 \leq \beta < 1) \). If

\[
0 < \sum_{i=1}^{n} \gamma_i \leq \frac{1}{1 - \beta},
\] (2.29)
Let $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}$ be the integral operator given by (1.5) and
\[ \left| \frac{z \mathcal{H}'_{\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)} + 1 \right| < 1. \] (2.30)

Then $\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$ is starlike of order $1 - (1 - \beta) \sum_{i=1}^{n} \gamma_i$.

Letting $n = 1$, $\gamma_1 = \gamma$ and $f_1 = f$ in Corollary 2.5, we get the following.

**Corollary 2.6.** Let $\gamma > 0$, and $f \in \Sigma_{\kappa_{\beta}} (-1 \leq \beta < 1)$. If
\[ 0 < \gamma \leq \frac{1}{1 - \beta}, \] (2.31)

$\mathcal{H}_{\gamma}(z)$ be the integral operator,
\[ \mathcal{H}_{\gamma}(z) = \frac{1}{z^2} \int_{0}^{z} \left( -u^2 f'(u) \right)^{\gamma} du, \] (2.32)

then $\mathcal{H}_{\gamma}(z)$ is starlike of order $1 - (1 - \beta) \gamma$.

Other work related to integral operator for different studies can also be found in [4–6].

**Acknowledgments**

The work here was supported by MOHE Grant: UKM-ST-06-FRGS0244-2010. The authors also would like to thank the referee for his/her careful reading and making some valuable comments which have improved the presentation of this paper.

**References**


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