Research Article

A Note on Some Properties of the Weighted $q$-Genocchi Numbers and Polynomials

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We consider the weighted $q$-Genocchi numbers and polynomials. From the construction of the weighted $q$-Genocchi numbers and polynomials, we investigate many interesting identities and relations satisfied by these new numbers and polynomials.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$, will, respectively, denote the ring of $p$-adic integers, the field, of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-\nu_p(p)} = 1/p$ (see [1–16]).

As well-known definition, the Euler numbers and Genocchi numbers are defined by

$$\frac{2}{e^t + 1} = e^{E_t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$ \hfill (1.1)

with the usual convention of replacing $E^n$ by $E_n$ and

$$\frac{2t}{e^t + 1} = e^{G_t} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!},$$ \hfill (1.2)
with the usual convention of replacing $G^n$ by $G_n$. We assume that $q \in \mathbb{C}$ with $|1 - q|^p < 1$ and that the $q$-number of $x$ is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$

(see [1–19]).

In [9], Kim introduced ordinary fermionic $p$-adic integral on $\mathbb{Z}_p$, and he studied some interesting relations and identities related to $q$-extension of Euler numbers and polynomials. In [8], he also introduced the $q$-extension of the ordinary fermionic $p$-adic integral on $\mathbb{Z}_p$ and he investigated many physical properties related to $q$-Euler numbers and polynomials. Recently, Kim firstly introduced the meaning of the weighted $q$-Euler number and polynomials by using the fermionic invariant $p$-adic integral on $\mathbb{Z}_p$ (see [14, 15]). In [16], Ryoo tried to study the weighted $q$-Euler number and polynomials by the same method of Kim et al. in [14] and the $q$-extension of the fermionic $p$-adic invariant integrals on $\mathbb{Z}_p$. As well-known properties, the Genocchi numbers are integers. The first few Genocchi numbers for $n = 2, 4, \ldots$ are $-1, 1, -3, 17, -155, 2073, \ldots$. The first few prime Genocchi numbers are $-3$ and $17$, which occur for $n = 6$ and $8$. There are no others with $n < 10^5$. These properties are very important to study in the area of fermionic distribution and $p$-adic numbers theory. By this reason, many mathematicians and physicians have studied Genocchi and Euler numbers which are in the different areas. By the same motivation, we consider weighted $q$-Genocchi polynomials and numbers by using the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ which are constructed by Kim and Ryoo (cf. [8, 16]).

In this paper, we consider the $q$-Genocchi numbers and polynomials with weighted $\alpha$ ($\alpha \in \mathbb{Q}$). From the construction of the weighted $q$-Genocchi numbers and polynomials, we investigate many interesting identities and relations satisfied by these new numbers and polynomials.

### 2. The Weighted $q$-Genocchi Numbers and Polynomials

Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions and, for $f \in \text{UD}(\mathbb{Z}_p)$, the fermionic $p$-adic invariant integral of $f$ on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \frac{1}{[pN]_{-1}} \sum_{x=0}^{pN-1} f(x)(-1)^x$$

(see [1–16]). If we take $f(x) = te^{xt}$, then we get

$$t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t + 1}.$$
By (1.2) and (2.2), we get

\[
\sum_{n=0}^{\infty} \frac{G_n}{n!} x^n = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) \frac{t^{n+1}}{n!} \\
= \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) \frac{t^{n+1}}{(n+1)!} \\
= \sum_{n=1}^{\infty} n \int_{\mathbb{Z}_p} x^{n-1} d\mu_{-1}(x) \frac{t^n}{n!}.
\]  

(2.3)

From (2.3),

\[
G_0 = 0, \quad \frac{G_n}{n} = \int_{\mathbb{Z}_p} x^{n-1} d\mu_{-1}(x), \quad n \in \mathbb{N}.
\]  

(2.4)

For \( f \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic \( q \)-integral of \( f \) on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p N]_{-q}^{-1}} \sum_{x=0}^{p^n-1} f(x) (-q)^x
\]  

(2.5)

(see [1–16]). From (2.5), we note that

\[
q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^l f(l),
\]  

(2.6)

where \( n \in \mathbb{N} \) and \( f_n(x) = f(x + n) \).

For \( \alpha \in \mathbb{Q} \), we consider the following fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \):

\[
t \int_{\mathbb{Z}_p} e^{[x]_{\alpha} t} d\mu_{-q}(x) = \sum_{n=0}^{\infty} \frac{G_{n,\alpha}}{n!} t^n,
\]  

(2.7)

where \( G_{n,\alpha} \) are called the \( n \)th \( q \)-Genocchi numbers with weight \( \alpha \). From (2.7), we get

\[
t \int_{\mathbb{Z}_p} e^{[x]_{\alpha} t} d\mu_{-q}(x) = t \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) \frac{t^{n+1}}{(n+1)!} \\
= \sum_{n=1}^{\infty} n \int_{\mathbb{Z}_p} [x]_{\alpha}^{n-1} d\mu_{-q}(x) \frac{t^n}{n!}.
\]  

(2.8)
By comparing the coefficients on the both sides of (2.7) and (2.8), we get

$$n \int_{\mathbb{Z}_p} [x]^{n-1} d\mu_q(x) = \tilde{G}^{(a)}_{n,q}, \quad n \in \mathbb{N}, \quad \tilde{G}^{(a)}_{0,q} = 0. \quad (2.9)$$

From (2.9), we obtain the following theorem.

**Theorem 2.1.** For $n \in \mathbb{N}$ and $a \in \mathbb{Q}$, one has

$$\int_{\mathbb{Z}_p} [x]^{n-1} d\mu_q(x) = \frac{\tilde{G}^{(a)}_{n,q}}{n}, \quad \tilde{G}^{(a)}_{0,q} = 0. \quad (2.10)$$

By the definition of fermionic $p$-adic $q$-integrals, we get

$$t \int_{\mathbb{Z}_p} [x]^{n-1} d\mu_q(x) = \frac{1}{(1 - q^a)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \int_{\mathbb{Z}_p} q^{alx} d\mu_q(x)$$

$$= \frac{[2]_q}{(1 - q^a)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1 + q^{al+1}}. \quad (2.11)$$

Therefore, we obtain the following theorem.

**Theorem 2.2.** For $n \in \mathbb{N}$ and $a \in \mathbb{Q}$, we have

$$\frac{\tilde{G}^{(a)}_{n,q}}{n} = \frac{[2]_q}{(1 - q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1 + q^{al+1}}. \quad (2.12)$$

By Theorem 2.2, we have the generating function of $\tilde{G}^{(a)}_{n,q}$ as follows:

$$\sum_{n=0}^{\infty} \tilde{G}^{(a)}_{n,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \frac{n}{(1 - q^a)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{alm+m} \frac{t^n}{n!}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{n}{(1 - q^a)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{alm} \frac{t^n}{n!}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{n}{(1 - q^m)^{n-1}} (1 - q^m)^{n-1} \frac{t^n}{n!}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{n [m]_{q^m}^{n-1} \frac{t^n}{n!}}{n!}.$$
Let $\tilde{F}_q^{(a)}(t)$ be the generating function of $\tilde{G}_{n,q}^{(a)}$. Then, by (2.9) and (2.13), we get

$$\tilde{F}_q^{(a)}(t) = [2]_q t \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_q e^t} = \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(a)} \frac{t^n}{n!}. \tag{2.14}$$

The $q$-Genocchi polynomials with weight $\alpha$ are defined by

$$\tilde{F}_q^{(a)}(t,x) = t \int_{\mathbb{Z}_q} e^{[x+y]_q e^t} d\mu_{-q}(y)$$

$$= \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(a)}(x) \frac{t^n}{n!}. \tag{2.15}$$

From (2.15), we get

$$t \int_{\mathbb{Z}_q} e^{[x+y]_q e^t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_q} [x + y]_q^n d\mu_{-q}(y) \frac{t^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{Z}_q} [x + y]_q^n d\mu_{-q}(y) \frac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} n \int_{\mathbb{Z}_q} [x + y]_q^{n-1} d\mu_{-q}(y) \frac{t^n}{n!}. \tag{2.16}$$

By (2.15) and (2.16), we obtain the following theorem.

**Theorem 2.3.** For $n \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$, one has

$$n \int_{\mathbb{Z}_q} [x + y]_q^{n-1} d\mu_{-q}(y) = \tilde{C}_{n,q}^{(a)}(x), \quad \tilde{C}_{0,q}^{(a)}(x) = 0. \tag{2.17}$$
We note that
\[
\int_{Z_p} [x + y]_{q^a}^{n-1} d\mu_q(y) = \sum_{l=0}^{n-1} \binom{n-1}{l} [x]_{q^a}^{n-1-l} q^{alx} \int_{Z_p} [y]_{q^a}^l d\mu_q(y)
\]
\[
= \sum_{l=0}^{n-1} \binom{n-1}{l} [x]_{q^a}^{n-1-l} q^{alx} \tilde{G}_{1+1,q}^{(a)}(l+1).
\]

(2.18)

From (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.4.** For \( n \in \mathbb{N} \) and \( a \in \mathbb{Q} \), one has

\[
\tilde{G}_{n,q}(x) = \frac{1}{n!} \sum_{l=0}^{n-1} \binom{n-1}{l} x_{q^a}^{n-1-l} q^{alx} \tilde{G}_{1+1,q}^{(a)}(l+1) \tag{2.19}
\]

From (2.15), we note that

\[
\tilde{F}_{q}^{(a)}(t, x) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} n \int_{Z_p} [x + y]_{q^a}^{n-1} d\mu_q(y) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{1}{q^{alx} - 1} \left(1 + q^{alx}\right) \frac{t^n}{n!}
\]
\[
= (2) \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{1}{q^{alx} - 1} \left(1 + q^{alx}\right) \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^m q^{a(n+m+1)} \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^m q^{a(n+m+1)} \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{1}{(1-q^n)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{a((x+m)+l)} \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{1}{(1-q^n)^{n-1}} \left(1 + q^{a(x+m)}\right) \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} \frac{1}{(1-q^n)^{n-1}} \left(1 - q^{a(x+m)}\right) \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} [x + m]_{q^a}^{n-1} \frac{t^n}{n!}
\]
\[
= (2) \sum_{m=0}^{\infty} (-1)^m q^m \sum_{n=0}^{\infty} [x + m]_{q^a}^{n-1} \frac{t^n}{(n-1)!}
\]
Therefore, we obtain the following theorem.

**Theorem 2.5.** For \( \alpha \in \mathbb{Q} \), one has

\[
\tilde{F}_q(t, x) = [2]_q t \sum_{m=0}^{\infty} (-1)^m q^m e^{[x+m]_q t}.
\]  

(2.21)

From (2.15) and (2.21), we obtain that

\[
\tilde{G}_{n, q}^{(a)}(x) = \frac{d^n}{dt^n} \tilde{F}_q(t, x) \bigg|_{t=0} = n[2]_q \sum_{m=0}^{\infty} (-1)^m q^m [x + m]_q^{m-1} \int
\]

\[
= n[2]_q \frac{1}{(1-q^n)} \sum_{l=0}^{n-1} \binom{n-1}{l} q^{alx} (-1)^l \frac{1}{1 + q^{al+1}} \]

\[
= \frac{n[2]_q}{(1-q^n)} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{alx} \frac{1}{1 + q^{al+1}}.
\]  

(2.22)

Therefore, we obtain the following theorem.

**Theorem 2.6.** For \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{Q} \), one has

\[
\tilde{G}_{n, q}^{(a)}(x) = \frac{n[2]_q}{(1-q^n)} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{alx} \frac{1}{1 + q^{al+1}}.
\]  

(2.23)

From (2.6), if we take \( f(x) = [x]_{q^n}^m = ((1-q^m)/(1-q^n))^m \), then we get

\[
q^n \int_{x_0}^x [x + n]_{q^n}^{m} d\mu_{-q}(x) = (-1)^n \int_{x_0}^x [x]_{q^n}^{m} d\mu_{-q}(x) + [2]_q m \sum_{l=0}^{n-1} (-1)^l q^l [l]_{q^n}^m.
\]  

(2.24)

By (2.17) and (2.24), we obtain the following theorem.
Theorem 2.7. For \( n \in \mathbb{N}, \ m \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \) and \( \alpha \in \mathbb{Q}, \) one has
\[
q^n \frac{\tilde{G}^{(a)}_{m+1,q}(n)}{m+1} = (-1)^n \frac{\tilde{G}^{(a)}_{m+1,q}}{m+1} + [2]_q \sum_{i=0}^{n-1} (-1)^i q^i [l]^m.
\] (2.25)

We remark that if we take \( n = 2s \ (s \in \mathbb{Z}_+) \) in Theorem 2.7, then we have
\[
q^{2s} \frac{\tilde{G}^{(a)}_{m+1,q}(2s)}{m+1} = \frac{\tilde{G}^{(a)}_{m+1,q}}{m+1} + [2]_q \sum_{i=0}^{2s-1} (-1)^i q^i [l]^m
\] (2.26)

and if we take \( n = 2s + 1 \ (s \in \mathbb{Z}_+) \) in Theorem 2.7, then we have
\[
q^{2s+1} \frac{\tilde{G}^{(a)}_{m+1,q}(2s + 1)}{m+1} + \frac{\tilde{G}^{(a)}_{m+1,q}}{m+1} = [2]_q \sum_{i=0}^{2s} (-1)^i q^i [l]^m.
\] (2.27)

From (2.27) with \( s = 0, \) we obtain the following corollary.

Corollary 2.8. For \( \alpha \in \mathbb{Q} \) and \( m \in \mathbb{Z}_+, \) one has
\[
q \frac{\tilde{G}^{(a)}_{m+1,q}(1)}{m+1} + \frac{\tilde{G}^{(a)}_{m+1,q}}{m+1} = \begin{cases} [2]_q & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}
\] (2.28)

From (2.19), we note that
\[
\frac{\tilde{G}^{(a)}_{m+1,q}(n)}{m+1} = \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) [x]_q^l \frac{\tilde{G}^{(a)}_{l+1,q}}{l+1} q^a l x = \frac{1}{m+1} \sum_{l=0}^{m} \left( 1 + \frac{l}{m+1} \right) [x]_q^l \frac{\tilde{G}^{(a)}_{l+1,q}}{l} q^a l x
\] (2.29)

\[
= \frac{1}{m+1} \sum_{l=1}^{m} \left( 1 + \frac{l}{m+1} \right) [x]_q^l \frac{\tilde{G}^{(a)}_{l,q}}{l} q^{a(l-1)x}
\]

\[
= \frac{1}{q^a m+1} \sum_{l=0}^{m} \left( 1 + \frac{l}{m+1} \right) [x]_q^l \frac{\tilde{G}^{(a)}_{l,q}}{l} q^{a l x}.
\]

From (2.29), we get
\[
q^a \tilde{G}^{(a)}_{m+1,q}(x) = \sum_{l=0}^{m+1} \left( \begin{array}{c} m+1 \\ l \end{array} \right) [x]_q^l \frac{\tilde{G}^{(a)}_{l,q}}{l} q^{a l x} = \left( [x]_q^a + q^a x \tilde{G}^{(a)}_{q} \right)^{m+1}.
\] (2.30)
with the usual convention about replacing \((G_{q}^{(a)})^n\) by \(\tilde{G}_{m,q}^{(a)}\). By (2.28) and (2.30), we get

\[
\frac{q^{1-a}q^a \tilde{G}_{m+1,q}^{(a)}(1)}{m + 1} + \frac{\tilde{G}_{m+1,q}^{(a)}}{m + 1} = \frac{q^{1-a}\left(1 + q^a \tilde{G}_{q}^{(a)}\right)^{m+1}}{m + 1} + \frac{\tilde{G}_{m+1,q}^{(a)}}{m + 1}.
\]  

From (2.28) and (2.31), we obtain the following theorem.

**Theorem 2.9.** For \(\alpha \in \mathbb{Q}\) and \(m \in \mathbb{Z}_+\), one has

\[
q^{1-a}\left(1 + q^a \tilde{G}_{q}^{(a)}\right)^{m+1} + \tilde{G}_{m+1,q}^{(a)} = \begin{cases} 
[2]_q & \text{if } m = 0, \\
0 & \text{if } m > 0.
\end{cases}
\]  

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**References**


