Research Article
Global Well-Posedness for a Family of MHD-Alpha-Like Models

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Global well-posedness is proved for a family of $n$-dimensional MHD-alpha-like models.

1. Introduction

In this paper, we consider a family of MHD-alpha-like models:

$$\partial_t v + (-\Delta)^{\theta} v + u \cdot \nabla u + \nabla \left( p + \frac{1}{2} b^2 \right) = b \cdot \nabla b, \quad (1.1)$$

$$\partial_t H + (-\Delta)^{\theta_1} H + u \cdot \nabla b - b \cdot \nabla u = 0, \quad (1.2)$$

$$v = \left[ 1 + \left( -a^2 \Delta \right)^{\theta} \right] u, \quad H = \left[ 1 + \left( -\alpha_M^2 \Delta \right)^{\theta_1} \right] b, \quad \alpha > 0, \alpha_M > 0, \quad (1.3)$$

$$\div v = \div u = \div H = \div b = 0, \quad (1.4)$$

$$(v, H)(0) = (v_0, H_0) \quad \text{in} \ \mathbb{R}^n(n \geq 3), \quad (1.5)$$

where $v$ is the fluid velocity field, $u$ is the “filtered” fluid velocity, $p$ is the pressure, $H$ is the magnetic field, and $b$ is the “filtered” magnetic field. $\alpha > 0$ and $\alpha_M > 0$ are the length scales and for simplicity we will take $\alpha = \alpha_M = 1$. The parameter $\theta_1 \geq 0$ affects
the strength of the nonlinear term and \( \theta_2 \geq 0 \) represents the degree of viscous dissipation satisfying

\[
3\theta_1 + 2\theta_2 = \frac{n + 2}{2}.
\] (1.6)

When \( \theta_1 = \theta_2 = 1 \) and \( n = 3 \), a global well-posedness is proved in [1]. The aim of this paper is to prove a global well-posedness theorem under (1.6). We will prove the following theorem.

**Theorem 1.1.** Let \( (u_0, b_0) \in H^s \) with \( s \geq 1 \), \( \text{div} \, v_0 = \text{div} \, u_0 = \text{div} \, H_0 = \text{div} \, b_0 = 0 \) in \( \mathbb{R}^n \), and (1.6) holding true. Then for any \( T > 0 \), there exists a unique strong solution \( (u, b) \) satisfying

\[
(u, b) \in L^\infty \left(0, T; H^{s+\theta_1}\right) \cap L^2 \left(0, T; H^{s+\theta_1+\theta_2}\right).
\] (1.7)

**Remark 1.2.** For studies on some standard MHD-\( \alpha \) or Leray-\( \alpha \) models, we refer to [2–7] and references therein.

**2. Proof of Theorem 1.1**

Since it is easy to prove that the problem (1.1)–(1.5) has a unique local smooth solution, we only need to establish the a priori estimates.

Testing (1.1) by \( u \), using (1.3) and (1.4), and letting \( \Lambda := (-\Delta)^{1/2} \), we see that

\[
\frac{1}{2} \frac{d}{dt} \int u^2 + \left| \Lambda^{\theta_1} u \right|^2 \, dx + \int \left| \Lambda^{\theta_2} u \right|^2 + \left| \Lambda^{\theta_1+\theta_2} u \right|^2 \, dx = \int (b \cdot \nabla) b \cdot u \, dx.
\] (2.1)

Testing (1.2) by \( b \) and using (1.3) and (1.4), we find that

\[
\frac{1}{2} \frac{d}{dt} \int b^2 + \left| \Lambda^{\theta_1} b \right|^2 \, dx + \int \left| \Lambda^{\theta_2} b \right|^2 + \left| \Lambda^{\theta_1+\theta_2} b \right|^2 \, dx = \int (b \cdot \nabla) u \cdot b \, dx.
\] (2.2)

Summing up (2.1) and (2.2), thanks to the cancellation of the right-hand side of (2.1) and (2.2), we infer that

\[
\frac{1}{2} \frac{d}{dt} \int (u, b)^2 + \left| \Lambda^{\theta_1} (u, b) \right|^2 \, dx + \int \left| \Lambda^{\theta_2} (u, b) \right|^2 + \left| \Lambda^{\theta_1+\theta_2} (u, b) \right|^2 \, dx = 0,
\] (2.3)

whence

\[
\|(u, b)\|_{L^2(0, T; H^{\theta_1+\theta_2})} \leq C.
\] (2.4)
Case 1. $\theta_1 + \theta_2 > 1$.

In the following calculations, we will use the following commutator estimates due to Kato and Ponce [8]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C \left( \|\nabla f\|_{L^p} \|\Lambda^{s-1} g\|_{L^p} + \|\Lambda^s f\|_{L^p} \| g\|_{L^q} \right), \quad (2.5)$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

We will also use the Sobolev inequality:

$$\|\nabla u\|_{L^p} \leq C \|\Lambda^{\theta_1 + \theta_2} u\|_{L^2} \left(1 - \frac{n}{p} = \theta_1 + \theta_2 - \frac{n}{2}\right), \quad (2.6)$$

and the Gagliardo-Nirenberg inequality:

$$\|\Lambda^s u\|_{L^{2p/(p-1)}}^2 \leq C \|\Lambda^{\theta_1 + \theta_2} u\|_{L^2} \|\Lambda^{\theta_1 + \theta_2} u\|_{L^2}. \quad (2.7)$$

Taking $\Lambda^s$ to (1.1), testing by $\Lambda^s u$, and using (1.3) and (1.4), we infer that

$$\frac{1}{2} \frac{d}{dt} \int |\Lambda^s u|^2 + |\Lambda^{s+\theta_1} u|^2 \, dx + \int |\Lambda^{s+\theta_2} u|^2 \, dx$$

$$= -\int [\Lambda^s(u \cdot \nabla u) - u \cdot \nabla \Lambda^s u] \Lambda^s u \, dx + \int [\Lambda^s(b \cdot \nabla b) - b \cdot \nabla \Lambda^s b] \Lambda^s u \, dx$$

$$+ \int b \cdot \nabla \Lambda^s b \cdot \Lambda^s u \, dx. \quad (2.8)$$

Taking $\Lambda^s$ to (1.2), testing by $\Lambda^s b$, and using (1.3) and (1.4), we deduce that

$$\frac{1}{2} \frac{d}{dt} \int |\Lambda^s b|^2 + |\Lambda^{s+\theta_1} b|^2 \, dx + \int |\Lambda^{s+\theta_2} b|^2 \, dx$$

$$= -\int [\Lambda^s(u \cdot \nabla b) - u \cdot \nabla \Lambda^s b] \Lambda^s b \, dx + \int [\Lambda^s(b \cdot \nabla u) - b \cdot \nabla \Lambda^s u] \Lambda^s b \, dx$$

$$+ \int b \cdot \nabla \Lambda^s u \cdot \Lambda^s b \, dx. \quad (2.9)$$
Summing up (2.8) and (2.9), thanks to the cancellation of the right-hand side of (2.8) and (2.9), and using (2.5), (2.6) and (2.7), we conclude that

\[
\frac{1}{2} \frac{d}{dt} \int |\Lambda^s(u, b)|^2 + |\Lambda^{s+\theta_1}(u, b)|^2 \, dx + \int |\Lambda^{s+\theta_2}(u, b)|^2 + |\Lambda^{s+\theta_1+\theta_2}(u, b)|^2 \, dx \\
\leq C \|\nabla u\|_{L^p}^2 \|\Lambda^s u\|_{L^{2p/(p-1)}}^2 + C \|\nabla b\|_{L^p} \|\Lambda^s b\|_{L^{2p/(p-1)}} \|\Lambda^s u\|_{L^{2p/(p-1)}} + C \|\nabla u\|_{L^p} \|\Lambda^s b\|_{L^{2p/(p-1)}}^2 \\
\leq C \|\nabla (u, b)\|_{L^p} \|\Lambda^s (u, b)\|_{L^{2p/(p-1)}}^2 \\
\leq C \|\Lambda^{s+\theta_1}(u, b)\|_{L^2} \|\Lambda^{s+\theta_2}(u, b)\|_{L^2} \|\Lambda^{s+\theta_1+\theta_2}(u, b)\|_{L^2} \\
\leq \frac{1}{2} \|\Lambda^{s+\theta_1+\theta_2}(u, b)\|^2_{L^2} + C \|\Lambda^{\theta_1}(v, H)\|^2_{L^2} \|\Lambda^{s+\theta_2}(u, b)\|^2_{L^2} \tag{2.10}
\]

which implies (1.7).

Case 2. \(0 < \theta_1 + \theta_2 \leq 1\) only when \(n = 3\).

Testing (1.1) by \(v\), using (1.4), we see that

\[
\frac{1}{2} \frac{d}{dt} \int v^2 \, dx + \int |\Lambda^s v|^2 \, dx = \int (b \cdot \nabla b - u \cdot \nabla u) v \, dx \\
\leq \left( \|b\|_{L^{p_1}} \|\nabla b\|_{L^{2p_1/(p_1-2)}} + \|u\|_{L^{p_1}} \|\nabla u\|_{L^{2p_1/(p_1-2)}} \right) \|v\|_{L^2} \\
\leq \|\nabla (u, b)\|_{L^{2p_1/(p_1-2)}} \|v\|_{L^2} \\
\leq C \|\nabla (u, b)\|_{H^{\theta_1+\theta_2}} \|\Lambda^{\theta_1}(v, H)\|_{L^2} \|v\|_{L^2}. \tag{2.11}
\]

Here we have used the Sobolev inequalities

\[
\|u\|_{L^{p_1}} \leq C \|u\|_{H^{\theta_1+\theta_2}} \left( -\frac{3}{p_1} = \theta_1 + \theta_2 - \frac{3}{2} \right), \\
\|\nabla (u, b)\|_{L^{2p_1/(p_1-2)}} \leq C \|\Lambda^{\theta_1}(v, H)\|_{L^2} \left( 1 - \frac{3(p_1-2)}{2p_1} = \theta_2 + 2\theta_1 - \frac{3}{2} \right). \tag{2.12}
\]

Similarly, testing (1.2) by \(H\) and using (1.4) and (2.12), we find that

\[
\frac{1}{2} \frac{d}{dt} \int H^2 \, dx + \int |\Lambda^s H|^2 \, dx = \int (b \cdot \nabla u - u \cdot \nabla b) H \, dx \\
\leq \|u\|_{L^{p_1}} \|\nabla (u, b)\|_{L^{2p_1/(p_1-2)}} \|H\|_{L^2} \tag{2.13}
\]

\[
\leq C \|u\|_{H^{\theta_1+\theta_2}} \|\Lambda^{\theta_1}(v, H)\|_{L^2} \|H\|_{L^2}.
\]

Combining (2.11) and (2.13) and using (2.4) and the Gronwall inequality, we have

\[
\|u\|_{L^2(0, T; H^{\theta_1+\theta_2})} \leq C. \tag{2.14}
\]
Similarly to (2.10), we have

\[
\frac{1}{2} \frac{d}{dt} \left[ |\Lambda^s(u, b)|^2 + |\Lambda^{s+\theta_1}(u, b)|^2 \right] dx + \int |\Lambda^{s+\theta_2}(u, b)|^2 dx \leq C \left\| \nabla (u, b) \right\|_{L^2} \left\| \Lambda^s(u, b) \right\|_{L^{2p_2/(p_2+1)}}^{2(1-\alpha_1)} \left\| \Lambda^{s+\theta_1}(u, b) \right\|_{L^2}^{2\alpha_1} \left\| \Lambda^{s+\theta_2}(u, b) \right\|_{L^2}^{2\alpha_1} 
\]

(2.15)

which implies (1.7) by \(1/(1 - \alpha_1) \leq 2\). Here we have used the Sobolev inequality:

\[
\left\| \nabla (u, b) \right\|_{L^2} \leq C \left\| (u, b) \right\|_{H^{0,2n}} \left( 1 - \frac{n}{p_2} < \theta_2 + 2\theta_1 - \frac{n}{2} \right) 
\]

(2.16)

and the Gagliardo-Nirenberg inequality:

\[
\left\| \Lambda^s(u, b) \right\|_{L^{2p_2/(p_2+1)}} \leq C \left\| \Lambda^{s+\theta_1}(u, b) \right\|_{L^2}^{1-\alpha_1} \left\| \Lambda^{s+\theta_2}(u, b) \right\|_{L^2}^{\alpha_1} 
\]

(2.17)

with \(-(p_2 - 1)/p_2)n = \alpha_1\theta_2 + \theta_1 - n/2\) and \(p_2 \geq 2 \geq 3/(2\theta_1 + \theta_2)\). This completes the proof.

**References**


