Research Article

Some New Fixed-Point Theorems for a \((\psi, \phi)\)-Pair Meir-Keeler-Type Set-Valued Contraction Map in Complete Metric Spaces

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We obtain some new fixed point theorems for a \((\psi, \phi)\)-pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski, (2007).

1. Introduction and Preliminaries

Let \((X, d)\) be a metric space, \(Y\) a subset of \(X\), and \(f : Y \to X\) a map. We say \(f\) is contractive if there exists \(\alpha \in [0, 1)\) such that, for all \(x, y \in Y\),

\[ d(fx, fy) \leq \alpha \cdot d(x, y). \]  

(1.1)

The well-known Banach’s fixed-point theorem asserts that if \(Y = X\), \(f\) is contractive and \((X, d)\) is complete, then \(f\) has a unique fixed point in \(X\). It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping \(f : X \to X\) is called a quasi-contraction if there exists \(k < 1\) such that

\[ d(fx, fy) \leq k \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \]  

(1.2)

for any \(x, y \in X\). In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed-point theorem.
Throughout this paper, by $\mathbb{R}$ we denote the set of all real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let $(X,d)$ be a metric space. Let $C(X)$ denote a collection of all nonempty closed subsets of $X$ and $CB(X)$ a collection of all nonempty closed and bounded subsets of $X$.

The existence of fixed points for various multivalued contractive mappings had been studied by many authors under different conditions. In 1969, Nadler Jr. [3] extended the famous Banach contraction principle from single-valued mapping to multivalued mapping and proved the below fixed-point theorem for multivalued contraction.

**Theorem 1.1 (see [3]).** Let $(X,d)$ be a complete metric space, and let $T$ be a mapping from $X$ into $CB(X)$. Assume that there exists $c \in [0,1)$ such that

$$d(Tx,Ty) \leq c d(x,y) \quad \forall x, y \in X,$$

where $d$ denotes the Hausdorff metric on $CB(X)$ induced by $d$; that is, $H(A,B) = \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}$, for all $A,B \in CB(X)$ and $d(x,B) = \inf_{z \in B} d(x,z)$. Then $T$ has a fixed point in $X$.

In 1989, Mizoguchi-Takahashi [4] proved the following fixed-point theorem.

**Theorem 1.2 (see [4]).** Let $(X,d)$ be a complete metric space, and let $T$ be a map from $X$ into $CB(X)$. Assume that

$$d(Tx,Ty) \leq \xi(d(x,y)) \cdot d(x,y),$$

for all $x,y \in X$, where $\xi : [0,\infty) \to [0,1)$ satisfies $\limsup_{s \to t^-} \xi(s) < 1$ for all $t \in [0,\infty)$. Then $T$ has a fixed point in $X$.

In 2006, Feng and Liu [5] gave the following theorem.

**Theorem 1.3 (see [5]).** Let $(X,d)$ be a complete metric space, and let $T : X \to C(X)$ be a multivalued map. If there exist $b,c \in (0,1)$, $c < b$ such that for any $x \in X$, there is $y \in T(x)$ satisfying the following two conditions:

(i) $b \cdot d(x,y) \leq D(x,Tx),$

(ii) $D(x,Ty) \leq c \cdot d(x,y).$

Then $T$ has a fixed point in $X$ provided that the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x,Tx)$, $x \in X$, is lower semicontinuous; that is, if for any $\{x_n\} \subset X$ and $x \in X$, $x_n \to x$, then $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

In 2007, Klim and Wardowski [6] proved the following fixed point theorem.

**Theorem 1.4 (see [6]).** Let $(X,d)$ be a complete metric space, and let $T : X \to C(X)$ be a multivalued map. Assume that the following conditions hold:

(i) the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x,Tx)$, $x \in X$, is lower semicontinuous;
(ii) there exist $b \in (0, 1)$ and $\varphi : [0, \infty) \to [0, b)$ such that

$$\forall\varepsilon \in [0, \infty), \quad \limsup_{r \to \varepsilon} \varphi(r) < b,$$

Then $T$ has a fixed point in $X$.

Recently, Pathak and Shahzad [7] introduced a new class of mapping $\Theta[0, A]$ and generalized the results of Klim and Wardowski [6]. Suppose that $A \in (0, \infty)$, $\Theta[0, A]$ denote the class of functions $\theta : [0, A) \to \mathbb{R}$ satisfying the following conditions:

1. $\theta$ is nondecreasing on $[0, A)$;
2. $\theta(t) > 0$ for all $t \in (0, A)$;
3. $\theta$ is subadditive in $(0, A)$; that is, $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$.

The following theorem was introduced in Pathak and Shahzad [7].

**Theorem 1.5** (see [7]). Let $(X, d)$ be a complete metric space and suppose that $T : X \to C(X)$. Assume that the following conditions hold:

(i) the mapping $f : X \to \mathbb{R}$ defined by $f(x) = D(x, Tx)$, $x \in X$, is lower semicontinuous,
(ii) there exists $\alpha : (0, \infty) \to (0, 1)$ such that

$$\forall \varepsilon \in [0, \infty), \quad \limsup_{r \to \varepsilon} \alpha(r) < 1,$$

(iii) there exists $\theta \in \Theta[0, A)$ satisfying the following condition:

$$\forall x \in X \quad \exists y \in T x \quad \{ \theta(d(x, y)) \leq \theta(D(x, Tx)) \},$$

$$\forall x \in X \quad \exists y \in T x \quad \{ \theta(D(y, Ty)) \leq \alpha(d(x, y)) \cdot \theta(d(x, y)) \}.$$  

Then $T$ has a fixed point in $X$.

Later, Kamran and Kiran [8] improved some results of Pathak and Shahzad [7] by allowing $T$ to have values in closed subsets of $X$. They proved that the function $\theta \in \Theta[0, A)$ is positive homogenous in $[0, A)$, that is,

4. $\theta(at) \leq a \theta(t)$ for all $a > 0$, $t \in [0, A)$,

and denote by $\Theta_h[0, A)$ the class of functions $\theta \in \Theta[0, A)$ satisfying condition (4). They proved the following theorem.

**Theorem 1.6** (see [8]). Let $(X, d)$ be a complete metric space and suppose that $\alpha$ is a function from $(0, \infty)$ to $[0, 1)$ such that

$$\forall \varepsilon \in [0, \infty), \quad \limsup_{r \to \varepsilon} \alpha(r) < 1.$$  

In this section, we first recall the notion of the Meir-Keeler-type function.

2. Main Results

Suppose that $T : X \to C(X)$. Assume that the following condition holds:

$$\theta(D(y, Ty)) \leq \alpha(d(x, y)) \cdot \theta(d(x, y)), \quad \text{for each } x \in X, \quad y \in Tx,$$

(1.9)

where $\theta \in \Theta_h(0, A)$. Then

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of $T$ and $\xi \in X$ such that $\lim_{n \to \infty} x_n = \xi$;

(ii) $\xi$ is a fixed point of $T$ if and only if the function $f(x) = D(x, Tx)$ is $T$-orbitally lower semicontinuous at $\xi$.

2.1. One calls $\psi : [0, \infty) \to [0, 1)$ the stronger Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $0 < t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new stronger Meir-Keeler-type function, as follows.

Definition 2.1. One calls $\psi : [0, \infty) \to [0, 1)$ the stronger Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \eta + \delta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t) < \gamma_\eta$.

Remark 2.2. It is clear that, if the function $\xi : [0, \infty) \to [0, 1)$ satisfies

$$\limsup_{s \to t} \xi(s) < 1$$

for all $t \in [0, \infty)$, then $\xi$ is also a stronger Meir-Keeler-type function.

Example 2.3. (1) If $\psi : [0, \infty) \to [0, 1), \psi(t) = k$ with $k \in (0, 1)$, then $\psi$ is a stronger Meir-Keeler-type function.

(2) If $\psi : [0, \infty) \to [0, 1), \psi(t) = t/(t+1)$, then $\psi$ is a stronger Meir-Keeler-type function.

Definition 2.4. Let $\psi : [0, \infty) \to [0, 1)$, $\phi : [0, \infty) \to [b, 1)$ be two functions where $0 < b < 1$. Then the mappings $\psi, \phi$ are called a $(\psi, \phi)$-pair Meir-Keeler-type function, if, for each $\eta > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $0 < t < \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\psi(t)/\phi(t) < \gamma_\eta$.

Remark 2.5. It is clear that if the functions $\psi : [0, \infty) \to [0, 1)$, $\phi : [0, \infty) \to [b, 1)$ satisfy

$$\limsup_{s \to t} \frac{\psi(s)}{\phi(s)} < 1,$$

(2.2)

for all $t \in [0, \infty)$, then $\phi, \psi$ are also a $(\psi, \phi)$-pair Meir-Keeler-type function.

Example 2.6. If $\psi : [0, \infty) \to [0, 1)$, $\psi(t) = t/(4t + 1)$ and $\phi : [0, \infty) \to [0, 1)$, $\phi(t) = t/(3t + 1)$, then $\phi, \psi$ are a $(\psi, \phi)$-pair Meir-Keeler-type function.
Definition 2.7. Let \((X, d)\) be a metric space, let \(\varphi : [0, \infty) \to [0, 1), \phi : [0, \infty) \to [b, 1)\) be two functions where \(0 < b < 1\), and let \(T : X \to 2^X\) be a set-valued map. Then \(T\) is called a \((\varphi, \phi)\)-pair Meir-Keeler-type set-valued contraction map, if the following conditions hold:

\begin{enumerate}[(C1)]
    
    \item for each \(\eta > 0\), there exists \(\delta > 0\) such that for \(x \in X\) with \(\eta \leq D(x, Tx) < \delta + \eta\), there exists \(\gamma \in [0, 1)\) such that
    \[
    \frac{\varphi(D(x, Tx))}{\phi(D(x, Tx))} < \gamma \eta
    \]

\end{enumerate}

\[
-C(2)\text{ for all } x \in X, \text{ there exists } y \in Tx \text{ such that}
\]

\[
\phi(D(x, Tx)) \cdot d(x, y) \leq D(x, Tx),
\]

\[
D(y, Ty) \leq \varphi(D(x, Tx)) \cdot d(x, y).
\]  

In this paper, we obtain some new fixed-point theorems for a \((\varphi, \phi)\)-pair Meir-Keeler-type set-valued contraction map in metric spaces. Our main results generalize and improve the results of Klim and Wardowski [6]. We now state our main theorem as follows.

Theorem 2.8. Let \((X, d)\) be a complete metric space, and let \(T : X \to C(X)\) be a \((\varphi, \phi)\)-pair Meir-Keeler-type set-valued contraction map. Then \(T\) has a fixed point in \(X\) provided the mapping \(f : X \to \mathbb{R}\) defined by \(f(x) = D(x, Tx), x \in X\), is lower semicontinuous.

Proof. Given \(x_0 \in X\) and by (C2), there exists \(x_1 \in X\) such that \(x_1 \in Tx_0\). Since \(T\) is a \((\varphi, \phi)\)-pair Meir-Keeler type set-valued contraction map, there exists \(x_1 \in Tx_0\) such that

\[
\phi(D(x_0, Tx_0)) \cdot d(x_0, x_1) \leq D(x_0, Tx_0),
\]

\[
D(x_1, Tx_1) \leq \varphi(D(x_0, Tx_0)) \cdot d(x_0, x_1).
\]

Continuing this process, we can choose a sequence \(\{x_n\} \subset X\) with \(x_{n+1} \in Tx_n\) such that, for all \(n \in \mathbb{N} \cup \{0\},

\[
\phi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}) \leq D(x_n, Tx_n),
\]

\[
D(x_{n+1}, Tx_{n+1}) \leq \varphi(D(x_n, Tx_n)) \cdot d(x_n, x_{n+1}).
\]

Therefore, we can deduce that, for all \(n \in \mathbb{N},

\[
D(x_{n+1}, Tx_{n+1}) \leq \frac{\varphi(D(x_n, Tx_n))}{\phi(D(x_n, Tx_n))} \cdot D(x_n, Tx_n)
\]

\[
< D(x_n, Tx_n).
\]
Thus, the sequence \( \{D(x_n, Tx_n)\}_{n=0}^{\infty} \) is decreasing and bounded below. Then there exists \( \eta \geq 0 \) such that

\[
\lim_{n \to \infty} D(x_n, Tx_n) = \eta. \tag{2.8}
\]

Hence, there exists \( \kappa_0 \in \mathbb{N} \) and \( \delta > 0 \) such that, for all \( n \geq \kappa_0 \),

\[
\eta \leq D(x_n, Tx_n) < \eta + \delta. \tag{2.9}
\]

By the condition (C1), we have that there exists \( \gamma_\eta \in [0, 1) \) such that

\[
\frac{\varphi (D(x_n, Tx_n))}{\phi (D(x_n, Tx_n))} < \gamma_\eta, \quad \forall n \geq \kappa_0. \tag{2.10}
\]

So for each \( n \in \mathbb{N} \) with \( n \geq \kappa_0 \), by (2.6), we can deduce that

\[
d(x_n, x_{n+1}) \leq \frac{\varphi (D(x_n, Tx_n))}{\phi (D(x_n, Tx_n))} \leq \frac{\varphi (D(x_{n-1}, Tx_{n-1}))}{\phi (D(x_{n-1}, Tx_{n-1}))} \cdot d(x_{n-1}, x_n) \leq \frac{\varphi (D(x_{n-1}, Tx_{n-1}))}{\phi (D(x_{n-1}, Tx_{n-1}))} \cdot \frac{D(x_{n-1}, Tx_{n-1})}{D(x_{n-1}, Tx_{n-1})} \leq \frac{1}{b} \cdot \frac{\varphi (D(x_{n-1}, Tx_{n-1}))}{\phi (D(x_{n-1}, Tx_{n-1}))} \cdot D(x_{n-1}, Tx_{n-1}) \leq \frac{1}{b} \cdot \gamma_\eta \cdot D(x_{n-1}, Tx_{n-1}) \leq \frac{1}{b} \cdot \gamma_\eta^2 \cdot D(x_{n-2}, Tx_{n-2}) \leq \frac{1}{b} \cdot \gamma_\eta^{n-1} \cdot D(x_{\kappa_0}, Tx_{\kappa_0}). \tag{2.11}
\]

Take \( m, n \in \mathbb{N} \) with \( m > n > \kappa_0 \). Then we get

\[
d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{1}{b} \cdot \frac{\gamma_\eta^{n-\kappa_0} \cdot D(x_{\kappa_0}, Tx_{\kappa_0})}{1 - \gamma_\eta}, \tag{2.12}
\]

and so we conclude that

\[
d(x_n, x_m) \to 0, \quad \text{as } m, n \to \infty, \tag{2.13}
\]
since $0 \leq \gamma_n < 1$. Thus, $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$.

Since $f : X \to \mathbb{R}$, $f(x) = d(x, Tx)$, $x \in X$, is lower semicontinuous, we have

$$0 \leq d(\mu, T\mu) = f(\mu) \leq \lim \inf_{n \to \infty} d(x_n, Tx_n) = 0. \quad (2.14)$$

The closeness of $T\mu$ implies $\mu \in T\mu$.

The following is a simple example for Theorem 2.8, and it generalize the result of Klim and Wardowski [6].

**Example 2.9.** Let $X = [0, 1]$ be a metric space with the standard metric $d$. Let $T : X \to C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{3} x^2 \right\}, \quad \forall x \in X. \quad (2.15)$$

Let $\psi : [0, \infty) \to [0, 1), \phi : [0, \infty) \to [2/3, 1)$ be defined by

$$\psi(t) = \frac{4}{9} + \frac{1}{9(t+1)}, \quad \phi(t) = \frac{2}{3} + \frac{1}{9(t+1)}, \quad \forall t \in [0, \infty). \quad (2.16)$$

Then $T$ is a $(\psi, \phi)$-pair Meir-Keeler-type set-valued contraction map, and $0 \in X$ is a fixed point of $T$.

In particular, if we let $\phi(t) = 2/3$, then this example satisfies all of the conditions of Theorem 1.4 (that was introduced in Klim and Wardowski [6]).

Using Example 3.1 in [6] and Example 1 in [10], we get the following another example for Theorem 2.8.

**Example 2.10.** Let $X = [0, 1]$ be a metric space with the standard metric $d$. Let $T : X \to C(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \left\{ \frac{1}{2} x^2 \right\}, & \text{if } x \in \left[ 0, \frac{15}{32} \right] \cup \left( \frac{15}{32}, 1 \right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{if } x = \frac{15}{32}. \end{cases} \quad (2.17)$$

Let $\psi : [0, \infty) \to [0, 1)$ be defined as in Example 1 of Cirić [10]:

$$\psi(t) = \begin{cases} \max \left\{ \frac{1}{12}, \frac{23}{12} t \right\}, & \text{if } t \in \left[ 0, \frac{1}{2} \right], \\ \left\{ \frac{23}{24} \right\}, & \text{if } t \in \left( \frac{1}{2}, \infty \right). \end{cases} \quad (2.18)$$
and let $\phi : [0, \infty) \to [1/12, 1)$ be defined by

$$
\phi(t) = \sqrt{\psi(t)} = \begin{cases} 
\max\left\{ \sqrt{\frac{T}{12}}, \sqrt{\frac{23T}{12}} \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\
\sqrt{\frac{23}{24}}, & \text{if } t \in \left(\frac{1}{2}, \infty\right).
\end{cases}
$$

(2.19)

Clearly, a function $f(x) = D(x, Tx)$ is lower semicontinuous. Then $\phi, \psi$ are a $(\psi, \phi)$-pair Meir-Keeler-type function, and $T$ is a $(\psi, \phi)$-pair Meir-Keeler-type set-valued contraction map. Moreover, by Theorem 2.8, we have that $0 \in X$ is a fixed point of $T$.

If we let $T : X \to C(X)$ be closed, then we also have the following fixed result.

**Theorem 2.11.** Let $(X, d)$ be a complete metric space, and let $T : X \to C(X)$ be a $(\psi, \phi)$-pair Meir-Keeler-type set-valued contraction map and closed. Then $T$ has a fixed point in $X$.

**Proof.** Following the proof of Theorem 2.8, we get that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$. Since $T$ is closed and $x_{n+1} \in Tx_n$, we have that $\mu \in T\mu$. \qed

The following is a simple example for Theorem 2.11.

**Example 2.12.** Let $X = [0, 1]$ be a metric space with the metric $d(x, y) := x$ for all $(x, y) \in X \times X$. Let $T : X \to C(X)$ be defined by

$$
T(x) = \left\{ \frac{1}{4} x^2 \right\}, \quad \forall x \in X.
$$

(2.21)

Let $\psi : [0, \infty) \to [0, 1), \phi : [0, \infty) \to [1/4, 1)$ be defined by

$$
\psi(t) = \frac{1}{4}, \quad \phi(t) = \frac{1}{2}, \quad \forall t \in [0, \infty).
$$

(2.22)

Then $T$ is a $(\psi, \phi)$-pair Meir-Keeler-type set-valued contraction map and closed, and $0 \in X$ is a fixed point of $T$.

Applying Theorem 2.8 and Remark 2.5, we are easy to get the following result.
**Theorem 2.13.** Let \((X,d)\) be a complete metric space, let \(\psi : [0, \infty) \to [0,1), \phi : [0, \infty) \to [b, 1)\) be two functions where \(0 < b < 1\), and let \(T : X \to C(X)\) be a set-valued contraction map. Suppose the following conditions hold:

1. for each \(t \in [0, \infty)\),
   \[
   \limsup_{s \to t} \frac{\psi(s)}{\phi(s)} < 1, \tag{2.23}
   \]

2. for all \(x \in X\), there exists \(y \in Tx\) such that
   \[
   \phi(D(x,Tx)) \cdot d(x,y) \leq D(x,Tx),
   \]
   \[
   D(y,Ty) \leq \psi(D(x,Tx)) \cdot d(x,y). \tag{2.24}
   \]

Then \(T\) has a fixed point in \(X\) provided the mapping \(f : X \to \mathbb{R}\) defined by \(f(x) = D(x,Tx), x \in X\), is lower semicontinuous.

The following is a simple example for Theorem 2.13.

**Example 2.14.** Let \(X = [0,1]\) be a metric space with the metric \(d, d(x,y) := x\) for all \((x,y) \in X \times Y\). Let \(T : X \to C(X)\) be defined as in Example 3.1 of Klim and Wardowski [6]:

\[
T(x) = \begin{cases} \left\{ \frac{1}{2} x^2 \right\}, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{if } x = \frac{15}{32}. \end{cases} \tag{2.25}
\]

Let \(\psi : [0, \infty) \to [0,1)\) be defined as in Example 1 of Ćirić [10]:

\[
\psi(t) = \begin{cases} \max\left\{ \frac{1}{12}, \frac{23}{12}, \frac{1}{2} \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left\{ \frac{23}{24} \right\}, & \text{if } t \in \left(\frac{1}{2}, \infty\right). \end{cases} \tag{2.26}
\]

and let \(\phi : [0, \infty) \to [1/12, 1)\) be defined by

\[
\phi(t) = \sqrt{\psi(t)}. \tag{2.27}
\]

Clearly, a function

\[
f(x) = D(x,Tx) = x \tag{2.28}
\]

is lower semicontinuous. Clearly, \(\limsup_{s \to t} \left(\frac{\psi(s)}{\phi(s)}\right) < 1\). We also conclude the following.
Case 1. If \( x \in [0, 15/32) \cup (15/32, 1] \), then \( y = T(x) = (1/2)x^2 \), and \( \psi, \phi \) satisfy the condition (2) of Theorem 2.13.

Case 2. If \( x = 15/32 \), then \( y = T(x) = 1/4 \) (resp., \( y = T(x) = 17/96 \)), and \( \psi, \phi \) also satisfy the condition (2) of Theorem 2.13.

Thus, by Theorem 2.13, we have that \( 0 \in X \) is a fixed point of \( T \).

Using Example 2.10, we also get the following example for Theorem 2.13.

Example 2.15. Let \( X = [0, 1] \) be a metric space with the standard metric \( d \). Let \( T : X \to C(X) \) be defined as

\[
T(x) = \begin{cases} 
\frac{1}{2} x^2, & \text{if } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\
\frac{17}{96} x^4, & \text{if } x = \frac{15}{32}.
\end{cases}
\] (2.29)

Let \( \psi : [0, \infty) \to [0, 1) \) be defined as

\[
\psi(t) = \begin{cases} 
\max \left\{ \frac{1}{12}, \frac{23}{12} t \right\}, & \text{if } t \in \left[0, \frac{1}{2}\right], \\
\frac{23}{24}, & \text{if } t \in \left(\frac{1}{2}, \infty\right),
\end{cases}
\] (2.30)

and \( \phi : [0, \infty) \to [1/12, 1) \) be defined by

\[
\phi(t) = \sqrt{\psi(t)}.
\] (2.31)

Clearly, \( \lim \sup_{s \to t^-} \psi(s)/\phi(s) < 1 \), and \( \psi, \phi \) satisfies all of the conditions of Theorem 2.13. So, we have that \( 0 \in X \) is a fixed point of \( T \).

If we let the function \( \phi : [0, \infty) \to [b, 1) \) be \( \phi(t) = b \) for all \( t \in [0, \infty) \) and let the function \( \psi : [0, \infty) \to [0, b) \), \( b \in (0, 1) \), be a stronger Meir-Keeler-type function; that is for if, for each \( \eta > 0 \), there exists \( \delta > 0 \) such that, for \( t \in [0, \infty) \) with \( \eta \leq t < \delta + \eta \), there exists \( \gamma_{\eta} \in [0, b) \) such that \( \psi(t) < \gamma_{\eta} \), then, by Theorem 2.8, it is easy to get the following theorem.

Theorem 2.16. Let \((X, d)\) be a complete metric space, let \( \psi : [0, \infty) \to [0, b) \), \( b \in (0, 1) \) be a stronger Meir-Keeler-type function, and let \( T : X \to C(X) \) be a set-valued contraction map. Suppose that, for all \( x \in X \), there exists \( y \in Tx \) such that

\[
b \cdot d(x, y) \leq D(x, Tx),
\]

\[
D(y, Ty) \leq \psi(D(x, Tx)) \cdot d(x, y).
\] (2.32)

Then \( T \) has a fixed point in \( X \) provided that the mapping \( f : X \to \mathbb{R} \) defined by \( f(x) = D(x, Tx) \), \( x \in X \), is lower semicontinuous.
The following is a simple example for Theorem 2.16.

**Example 2.17.** Let $X = [0, 1]$ be a metric space with the standard metric $d$. Let $T : X \rightarrow C(X)$ be defined by

$$T(x) = \left\{ \frac{1}{3} x^2 \right\}, \quad \forall x \in X. \quad (2.33)$$

Let $\psi : [0, \infty) \rightarrow [0, 2/3)$ be defined by

$$\psi(t) = \frac{4}{9} + \frac{1}{9(t + 1)}, \quad \forall t \in [0, \infty). \quad (2.34)$$

Then $\psi$ a stronger Meir-Keeler-type function, and $0 \in X$ is a fixed point of $T$.

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**References**


