Research Article

Existence Theory for Integrodifferential Equations and Henstock-Kurzweil Integral in Banach Spaces

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We prove existence theorems for the integrodifferential equation

\[ x'(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds), \quad x(0) = x_0, \quad t \in I_a = [0, a], \quad a > 0, \]

where the integral is taken in the sense of HL \[1\].

The Henstock-Kurzweil integral encompasses the Newton, Riemann, and Lebesgue integrals \[2–5\]. A particular feature of this integral is that integrals of highly oscillating functions such as \( F'(t) \), where \( F(t) = t^2 \sin t^{-2} \) on \((0, 1]\) and \( F(0) = 0 \), can be defined. This
integral was introduced by Henstock and Kurzweil independently in 1957-1958 and has since proved useful in the study of ordinary differential equations [6–10].

It is well known that Henstock’s lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach space-valued functions, Cao pointed out in [1] that Henstock’s lemma holds for the case of finite dimension, but it does not always hold for the case of infinite dimension.

In this paper, we will use the definition of HL integral which satisfies Henstock’s lemma.

As each Bochner integrable function is HL integrable function, so our results extend corresponding theorems for differential, integral, and integродifferential equations where the right-hand side of (1.1) was Bochner integrable. We obtain this result even using Carathéodory concept of a solution and not the approximate derivative, which comes from the properties of primitives of Henstock-Kurzweil-Denjoy concept of the integral.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1.1) (see, e.g., [11–17]).

A Mönch fixed point theorem [18] and the techniques of the theory of the measure of noncompactness are used to prove the existence of solution of problem (1.1). By using some conditions expressed in terms of the measure of noncompactness which the function \( f \) satisfies, we define a completely continuous operator \( F \) over the Banach space \( C([0,a]) \), whose fixed points are solutions of (1.1). The fixed point theorem of Mönch is used to prove the existence of a fixed point of the operator \( F \).

Our fundamental tools are the Kuratowski and Hausdorff measures of noncompactness [19].

For any bounded subset \( A \) of \( E \) we denote by \( \alpha(A) \) the Kuratowski measure of noncompactness of \( A \), that is, the infimum of all \( \varepsilon > 0 \), such that there exists a finite covering of \( A \) by sets of diameter smaller than \( \varepsilon \).

For any bounded subset \( A \) of \( E \) we denote by \( \alpha_1(A) \) the Hausdorff measure of noncompactness of \( A \), that is, the infimum of all \( \varepsilon > 0 \), such that \( A \) can be covered by a finite number of balls of radius smaller than \( \varepsilon \).

The properties of measure of noncompactness \( \gamma = \alpha \) and \( \gamma = \alpha_1 \) are as follows:

(i) if \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \);
(ii) \( \gamma(A) = \gamma(\bar{A}) \), where \( \bar{A} \) denotes the closure of \( A \);
(iii) \( \gamma(A) = 0 \) if and only if \( A \) is relatively compact;
(iv) \( \gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\} \);
(v) \( \gamma(\lambda A) = |\lambda| \gamma(A), (\lambda \in \mathbb{R}) \);
(vi) \( \gamma(A + B) \leq \gamma(A) + \gamma(B) \);
(vii) \( \gamma(\text{conv } A) = \gamma(A) \);
(viii) \( \alpha_1(A) \leq \alpha(A) \leq 2\alpha_1(A) \).

**Lemma 1.1** [20]. Let \( H \subset C(I_a,E) \) be a family of strongly equicontinuous functions. Let, for \( t \in I_a \), \( H(t) = \{h(t) \in E, h \in H\} \). Then \( \alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a)) \), where \( \alpha_C(H) \) denotes the measure of noncompactness in \( C(I_a,E) \) and the function \( t \mapsto \alpha(H(t)) \) is continuous.
We now gather some well-known definitions and results from the literature, which we
will use throughout this paper.

**Definition 1.2.** A function \( f : I_a \times E \times E \to E \) is \( L^1 \)-Carathéodory if the following conditions hold:
(i) the map \( t \to f(t,x,y) \) is measurable for all \((x,y) \in E^2\);
(ii) the map \((x,y) \to f(t,x,y)\) is continuous for almost all \( t \in I_a \).

**Definition 1.3.** A function \( k : I_a \times I_a \times E \to E \) is \( L^1 \)-Carathéodory if the following conditions hold:
(i) the map \((t,s) \to k(t,s,y)\) is measurable for all \( y \in E \);
(ii) the map \( y \to k(t,s,y) \) is continuous for almost all \((t,s) \in I_a^2\).

**Definition 1.4.** A nonnegative real-valued function \((t,s,z) \to h(t,s,z)\) defined on \( I_a \times I_a \times E \) is a Kamke function if \( h \) satisfies the Carathéodory conditions, and for each fixed \( t, s \), the function \( z \to h(t,s,z) \) is nondecreasing and for each \( q, 0 < q \leq a \), the function identically equal to zero is the unique continuous solution of the integral equation \( z(t) = \int_0^t h(t,s,z(s))ds \) defined on \([0,q]\).

**Definition 1.5** [3, 5]. A family \( \mathcal{F} \) of functions \( F \) is said to be uniformly absolutely continuous in the restricted sense on \( A \subseteq [a,b] \) or in short uniformly \( AC_\#(A) \) if for every \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that for every \( F \) in \( \mathcal{F} \) and for every finite or infinite sequence of nonoverlapping intervals \([a_i,b_i]\) with \( a_i,b_i \in A \) and satisfying \( \sum_i |b_i - a_i| < \eta \), then \( \sum_i \omega(F,[a_i,b_i]) < \varepsilon \), where \( \omega \) denotes the oscillation of \( F \) over \([a_i,b_i]\) (i.e., \( \omega(F,[a_i,b_i]) = \sup\{|F(r) - F(s)| : r,s \in [a_i,b_i]\} \)).

A family \( \mathcal{F} \) of functions \( F \) is said to be uniformly generalized absolutely continuous in the restricted sense on \([a,b]\) or uniformly \( ACG_\# \) if \([a,b]\) is the union of a sequence of closed sets \( A_i \) such that on each \( A_i \), the function \( F \) is uniformly \( AC_\#(A_i) \).

In the proof of the main theorem we will apply the following fixed point theorem.

**Theorem 1.6** [18]. Let \( D \) be a closed convex subset of \( E \), and let \( F \) be a continuous map from \( D \) into itself. If for some \( x \in D \), the implication
\[ \overline{V} = \text{conv}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively compact} \]
holds for every countable subset \( V \) of \( D \), then \( F \) has a fixed point.

2. Henstock-Kurzweil integral in Banach spaces

In this part, we present the Henstock-Kurzweil integral in a Banach space and we give some properties of this integral.

**Definition 2.1.** Let \( \delta \) be a positive function defined on the interval \([a,b]\). A tagged interval \((x, [c,d])\) consists of an interval \([c,d] \subset [a,b] \) and a point \( x \in [c,d] \). The tagged interval \((x, [c,d])\) is subordinate to \( \delta \) if \([c,d] \subset [x - \delta(x), x + \delta(x)] \).
The letter \( P \) will be used to denote finite collections of nonoverlapping tagged intervals. Let 

\[
P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n\}, \quad n \in \mathbb{N},
\]

be such a collection in \([a, b]\). Then

(i) the points \( \{s_i : 1 \leq i \leq n\} \) are called the tags of \( P \),
(ii) the intervals \( \{[c_i, d_i] : 1 \leq i \leq n\} \) are called the intervals of \( P \),
(iii) if \((s_i, [c_i, d_i])\) is subordinate to \( \delta \) for each \( i \), then we write \( P \) is sub-\( \delta \),
(iv) if \([a, b] = \bigcup_{i=1}^{n} [c_i, d_i] \), then \( P \) is called a tagged partition of \([a, b]\),
(v) if \( P \) is a tagged partition of \([a, b]\) and if \( P \) is sub-\( \delta \), then we write \( P \) is sub-\( \delta \) on \([a, b]\).

If \( f : [a, b] \to E \), then \( F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} (F(d_i) - F(c_i)). \) For such a function, \( F(P) = F(b) - F(a) \) if \( P \) is a tagged partition of \([a, b]\).

**Definition 2.2** [1]. A function \( f : [a, b] \to E \) is **Henstock-Kurzweil integrable on** \([a, b]\) \((f \in \text{HK}([a, b], E))\) if there exists a vector \( z \) in \( E \) with the following property: for each \( \varepsilon > 0 \), there exists a positive function \( \delta \) on \([a, b]\) such that \( \|f(P) - z\| < \varepsilon \) whenever \( P \) is sub-\( \delta \) on \([a, b]\). The function \( f \) is **Henstock-Kurzweil integrable on a measurable set** \( A \subset [a, b] \) if \( f \chi_A \) is Henstock-Kurzweil integrable on \([a, b]\). The vector \( z \) is the Henstock-Kurzweil integral of \( f \).

We note that this definition includes the generalized Riemann integral defined by Gordon in [21].

**Definition 2.3** [1]. A function \( f : [a, b] \to E \) is **HL integrable on** \([a, b]\) \((f \in \text{HL}([a, b], E))\) if there exists a function \( F : [a, b] \to E \), defined on the interval \([a, b]\), satisfying the following property: given \( \varepsilon > 0 \), there exists a positive function \( \delta \) on \([a, b]\) such that if \( P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n\} \) is a tagged partition of \([a, b]\) sub-\( \delta \), then

\[
\sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon.
\]

We note that by the triangle inequality, \( f \in \text{HL}([a, b], E) \) implies \( f \in \text{HK}([a, b], E) \). In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

**Definition 2.4.** Let \( f : I_a \to E \) be Henstock-Kurzweil integrable on \([a, b]\). Then a function \( F(t) = \int_{a}^{t} f(s) \, ds \), which is defined on subintervals of \([a, b]\) and the integral is in the sense of Henstock-Kurzweil, is called the **primitive of** \( f \).

For the integral, we have the following theorems.
Theorem 2.5 [1]. Let \( f : [a, b] \to E \). If \( f = 0 \) almost everywhere on \([a, b]\), then \( f \) is HL integrable on \([a, b]\) and \( \int_a^b f(t)dt = 0 \).

Theorem 2.6 [1]. Let \( f : [a, b] \to E \) be HL integrable on \([a, b]\) and let \( F(x) = \int_a^x f(t)dt \) for each \( x \in [a, b] \). Then
(i) the function \( F \) is continuous on \([a, b]\),
(ii) the function \( F \) is differentiable almost everywhere on \([a, b]\) and \( F' = f \),
(iii) \( f \) is measurable.

Theorem 2.7 (see [22, Theorem 5]). Suppose that \( f_n : [a, b] \to E \), \( n = 1, 2, \ldots \), is a sequence of HL integrable functions satisfying the following conditions:
(i) \( f_n(x) \to f(x) \) almost everywhere in \([a, b]\), as \( n \to \infty \);
(ii) the set of primitives of \( f_n \), \( \{F_n(x)\} \), where \( F_n(x) = \int_a^x f_n(s)ds \), is uniformly ACG* in \( n \);
(iii) the primitives \( F_n \) are equicontinuous on \([a, b]\),
then \( f \) is HL integrable on \([a, b]\) and \( \int_a^x f_n \to \int_a^x f \) uniformly on \([a, b]\), as \( n \to \infty \).

We note that this theorem for Denjoy-Bochner integrals is mentioned in [22] without a proof. It is also true for HL integrals. The proof is similar to that of [5, Theorem 7.6], see also [23, Theorem 4].

Theorem 2.8. If the function \( f : I_a \to E \) is HL integrable, then
\[
\int_I f(t)dt \in |I| \cdot \text{conv} f(I),
\]
where \( \text{conv} f(I) \) is the closure of the convex of \( f(I) \), \( I \) is an arbitrary subinterval of \( I_a \), and \( |I| \) is the length of \( I \).

The proof is similar to that of [24, Lemma 2.1.3]; see also [25, Theorem 10.4, page 268].

Lemma 2.9 [10]. Let \( E_1 \) be a separable Banach space. Suppose that \( V \) is a countable set of HL integrable functions. Let \( F = \{ \int_0^t x(s)ds, x \in V, t \in I_a \} \) be an equicontinuous, equibounded, and uniformly ACG* on \( I_a \). Then \( \alpha_1(\int_0^t V(s)ds) \leq \int_0^t \alpha_1(V(s))ds, t \in I_a \), whenever \( \alpha_1(V(s)) \leq \varphi(s) \) for \( s \in I_a \) a.e. \( \varphi \) is a Lebesgue integrable function and \( \alpha_1 \) denotes the Hausdorff measure of noncompactness.

3. Existence results for integrodifferential equations

We will consider the problem
\[
x(t) = x_0 + \int_0^t f \left( z, x(z), \int_0^z k(z, s, x(s))ds \right)dz, \quad t \in I_a, x_0 \in E,
\]
where integrals are taken in the sense of HL.

To obtain the existence result it is necessary to define a notion of a solution.
Definition 3.1. An $ACG_*$ function $x : I_a \rightarrow E$ is said to be a solution of problem (1.1) if it satisfies the following conditions:

(i) $x(0) = x_0$;

(ii) $x'(t) = f(t,x(t), \int_0^t k(t,s,x(s))ds)$ for a.e. $t \in I_a$.

Definition 3.2. A continuous function $x : I_a \rightarrow E$ is said to be a solution of the problem (3.1) if it satisfies $x(t) = x_0 + \int_0^t f(z,x(z), \int_0^z k(z,s,x(s))ds)dz$ for every $t \in I_a$.

Let us observe that each solution $x$ of problem (1.1) is equivalent to the solution of problem (3.1).

Let $x$ be a continuous solution of (1.1). By definition, $x$ is $ACG_*$ function and $x(0) = x_0$. Since, for a.e. $t \in I_a$, $x'(t) = f(t,x(t), \int_0^t k(t,s,x(s))ds)$ and the last is HL integrable, so is differentiable a.e. Moreover, $\int_0^t f(z,x(z), \int_0^z k(z,s,x(s))ds)dz = \int_0^t x'(s)ds = x(t) - x_0$. Thus satisfies (3.1).

Now assume that $y$ is $ACG_*$ function and it is clear that $y(0) = x_0$. By the definition of HL integrals there exists an $ACG_*$ function $G$ such that $G(0) = x_0$ and $G'(t) = f(t,y(t), \int_0^t k(t,s,y(s))ds)$ a.e. Hence

$$y(t) = x_0 + \int_0^t f(z,y(z), \int_0^z k(z,s,y(s))ds)dz = x_0 + \int_0^t G'(s)ds = x_0 + G(t) - G(0) = G(t).$$

We obtain $y = G$ and then $y'(t) = f(t,y(t), \int_0^t k(t,s,y(s))ds)$.

For $x \in C(I_a,E)$, we define the norm of $x$ by $\|x\|_C = \sup\{\|x(t)\|, t \in I_a\}$.

Now we present an existence theorem for the problem (1.1) in a separable Banach space $E_1$.

Let $B = \{x \in C(I_a,E) : \|x\|_C \leq \|x_0\|_C + p, \ p > 0\}$. Note that this set is closed and convex.

Define the operator $F : C(I_a,E) \rightarrow C(I_a,E)$ by

$$F(x)(t) = x_0 + \int_0^t f(z,x(z), \int_0^z k(z,s,x(s))ds)dz, \ t \in I_a, \ x_0 \in E. \ (3.3)$$

Let

$$\Gamma = \{F(x) \in C(I_a,E) : x \in B\}. \ (3.4)$$

Theorem 3.3. Assume that, for each $ACG_*$ function $x : I_a \rightarrow E_1$, functions $k(\cdot, s, x(s))$, $f(\cdot, x(\cdot), \int_0^t k(\cdot, s, x(s))ds)$ are HL integrable, $f$ and $k$ are $L^1$-Carathéodory functions. Suppose that there exists a constant $d$ such that

$$\alpha(f(t,A,C)) \leq d \cdot \max\{\alpha(A),\alpha(C)\} \quad \text{for each bounded subset } A, C \subset E_1, \ t \in I_a, \ (3.5)$$

where $\alpha$ denotes the Kuratowski measure of noncompactness.
Assume that

$$\alpha(k(t,s,X)) \leq h(t,s,\alpha(X)) \quad \text{for each bounded subset } X \subset E_1, \ 0 \leq s \leq t \leq a,$$

(3.6)

where $h$ is a Kamke function.

Moreover, let $F$ be equicontinuous, equibounded, and uniformly $\text{ACG}_*$ on $I_a$. Then there exists at least one solution of problem (1.1) on $I_c$ for some $0 < c \leq a$ and $d \cdot c < 1$.

Proof. By equicontinuity and equiboundedness of $\Gamma$, there exists a number $c, 0 < c \leq a$, such that

$$\left\| \int_0^t f \left( z, x(z), \int_0^z k(z,s,x(s)) \, ds \right) \, dz \right\| \leq p \quad \text{for fixed } p > 0, \ t \in I_c, \ x \in B. \quad (3.7)$$

By our assumptions the operator $F$ is well defined and maps $B$ into $B$,

$$\|F(x)(t)\| = \|x_0 + \int_0^t f \left( z, x(z), \int_0^z k(z,s,x(s)) \, ds \right) \, dz\| \leq \|x_0\|_C + \int_0^t \left\| f \left( z, x(z), \int_0^z k(z,s,x(s)) \, ds \right) \right\| \, dz \leq \|x_0\|_C + p. \quad (3.8)$$

Using Theorem 2.6 we deduce that $F$ is continuous.

Suppose that $V \subset B$ satisfies the condition $V = \text{conv}(\{x\} \cup F(V))$ for some $x \in B$. We will prove that $V$ is relatively compact, thus (1.2) is satisfied. Theorem 1.6 will ensure that $F$ has a fixed point.

Let, for $t \in I_c$, $V(t) = \{v(t) \in E_1 : v \in V\}$. Since $V$ is equicontinuous, so by Lemma 1.1, $t \mapsto v(t) = \alpha(V(t))$ is continuous on $I_c$.

For fixed $t \in I_c$, we divide the interval $[0, t]$ into $m$ parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, \ldots, m$. We denote $T_i = [t_i, t_{i+1}], i = 0, 1, \ldots, m - 1$. Let us fix $z \in I_c$. Let $\int_0^z K(s) \, ds = \{\int_0^z x(s) : x \in K\}$ for any $K \subset C(I_c, E_1)$ and let $\tilde{k}_z$ denotes the mapping defined by $\tilde{k}_z(x(s)) = k(z,s,x(s))$ for each $x \in B_1$ and $s \in I_c$. Obviously, $\tilde{k}_z(V(s)) = k(z,s,V(s))$.

Let

$$F(V(t)) = \{F(x)(t) \in C(I_c, E_1) : x \in V, t \in I_c\}
= \left\{x_0 + \int_0^t f \left( z, x(z), \int_0^z k(z,s,x(s)) \, ds \right) \, dz : x \in V, t \in I_c \right\}. \quad (3.9)$$

By Theorem 2.8 and the properties of the HL integral, we have

$$F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f \left( z, x(z), \int_0^z k(z,s,x(s)) \, ds \right) \, dz
\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \text{conv} f \left( z, V(T_i), \int_0^z \tilde{k}_z(V(s)) \, ds \right) \quad \text{for each } x \in V. \quad (3.10)$$

Therefore, $F(V(t)) \subset x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \text{conv} f \left( z, V(T_i), \int_0^z \tilde{k}_z(V(s)) \, ds \right)$. 
Using (3.5), (3.6), Lemma 2.9, and the properties of the measure of noncompactness $\alpha$ we have

$$\alpha(F(V(t))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \alpha \left( f \left( z, V(T_i), \int_0^z \tilde{k}_z(V(s))\,ds \right) \right)$$

$$\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d \cdot \max \left\{ \alpha(V(T_i)), \alpha \left( \int_0^z \tilde{k}_z(V(s))\,ds \right) \right\}.$$  \hspace{1cm} (3.11)

Let us observe that

(i) if $\alpha(V(T_i)) > \alpha(\int_0^z \tilde{k}_z(V(s))\,ds)$, then

$$\alpha(V) = \alpha(\overline{\text{conv}}(\{x\} \cup F(V))) \leq \alpha(F(V)) < d \cdot c \cdot \alpha(V),$$  \hspace{1cm} (3.12)

because $d \cdot c < 1$, so $\alpha(V) < \alpha(V)$, a contradiction;

(ii) if $\alpha(V(T_i)) < \alpha(\int_0^z \tilde{k}_z(V(s))\,ds)$, then

$$\alpha(V) < \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d \cdot \alpha \left( \int_0^z \tilde{k}_z(V(s))\,ds \right) \leq 2dc\alpha_1 \left( \int_0^z \tilde{k}_z(V(s))\,ds \right)$$

$$\leq 2dc \int_0^z \alpha_1(k(z,s,V(s))\,ds) \leq 2dc \int_0^z \alpha(k(z,s,V(s))\,ds) \leq 2dc \int_0^z h(z,s,V(s))\,ds,$$  \hspace{1cm} (3.13)

since $V = \overline{\text{conv}}(\{x\} \cup F(V))$ so

$$v(t) \leq 2dc \int_0^z h(z,s,v(s))\,ds.$$  \hspace{1cm} (3.14)

Hence applying now a theorem on differential inequalities we get $v(t) = \alpha(V(t)) = 0$. By Arzelà–Ascoli theorem, $V$ is relatively compact. So, by Theorem 1.6, $F$ has a fixed point which is a solution of the problem (1.1). \hfill \Box

For real-valued Banach space $E$, we have the following theorem.

**Theorem 3.4.** Assume that for each $ACG_e$ function $x : I_\alpha \to E$, functions $k(\cdot,s,x(s)), f(\cdot,s), x(\cdot), \int_0^{\cdot} k(\cdot,s,x(s))\,ds$ are HL integrable, $f$ and $k$ are $L^1$-Carathéodory functions. Suppose that there exists a constant $d_1$ such that

$$\alpha \left( f(t,A,C) \right) \leq d_1 \cdot \max \{ \alpha(A), \alpha(C) \} \quad \text{for each bounded subset } A, C \subset E, \quad t \in I_\alpha.$$  \hspace{1cm} (3.15)

Assume that there exists a continuous function $d_2 : I_\alpha \times I_\alpha \to R_+$ such that

$$\alpha(k(I,I,X)) \leq \sup_{s \in I} d_2(t,s)\alpha(X) \quad \text{for each bounded subset } X \subset E, \quad t,s \in I, \quad I \subset I_\alpha,$$  \hspace{1cm} (3.16)
and the zero function is the unique continuous solution of the inequality
\[ p(t) \leq d_1 \cdot c \cdot \sup_{z \in I_a} \int_0^c d_2(z, s) p(s) ds \quad \text{on } I_c. \]  
(3.17)

Moreover, let \( \Gamma \) be equicontinuous, equibounded, and uniformly ACG\(_*\) on \( I_a \). Then there exists a solution of problem (1.1) on \( I_c \) for some \( 0 < c \leq a \) and \( d_1 \cdot c < 1 \).

**Proof.** By equicontinuity and equiboundedness of \( \Gamma \), there exists a number \( c, 0 < c \leq a \), such that
\[ \left\| \int_0^t f \left( z, x(z), \int_0^z k(z, s, x(s)) ds \right) dz \right\| \leq p \quad \text{for fixed } p > 0, x \in B, t \in I_c. \]  
(3.18)

By our assumptions the operator \( F \) is well defined and maps \( B \) into \( B \),
\[ \left\| F(x)(t) \right\| = \left\| x_0 + \int_0^t f \left( z, x(z), \int_0^z k(z, s, x(s)) ds \right) dz \right\| \leq \left\| x_0 \right\|_C + \int_0^t \left\| f \left( z, x(z), \int_0^z k(z, s, x(s)) ds \right) dz \right\| \leq \left\| x_0 \right\|_C + l. \]  
(3.19)

Using Theorem 2.6 we deduce that \( F \) is continuous.

Suppose that \( V \subset B \) satisfies the condition \( \overline{V} = \overline{\text{conv}}(\{x \} \cup F(V)) \) for some \( x \in B \). We will prove that \( V \) is relatively compact, thus (1.2) is satisfied. Theorem 1.6 will ensure that \( F \) has a fixed point.

Let, for \( t \in I_c \), \( V(t) = \{ v(t) \in E : v \in V \} \). Since \( V \) is equicontinuous, so by Lemma 1.1, \( t \mapsto v(t) = \alpha(V(t)) \) is continuous on \( I_c \).

For fixed \( t \in I_c \) we divide the interval \( [0, t] \) into \( m \) parts: \( 0 = t_0 < t_1 < \cdots < t_m = t \) where \( t_i = it/m, i = 0, 1, \ldots, m \), and for fixed \( z \in [0, t] \) we divide the interval \( [0, z] \) into \( m \) parts: \( 0 = z_0 < z_1 < \cdots < z_m = z \), where \( z_j = jz/m, j = 0, 1, \ldots, m \).

Let \( V([z_j, z_{j+1}]) = \{ u(s) : u \in V, z_j \leq s \leq z_{j+1} \}, j = 0, 1, \ldots, m - 1 \). By Lemma 1.1 and the continuity of \( v \) there exists \( s_j \in I_j = [z_j, z_{j+1}] \) such that
\[ \alpha(V([z_j, z_{j+1}])) = \sup \{ \alpha(V(s)) : z_j \leq s \leq z_{j+1} \} = v(s_j). \]  
(3.20)

By Theorem 2.8 and the properties of the HL integral we have
\[ F(u)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f \left( z, u(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k(z, s, u(s)) ds \right) dz \]
\[ \in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\text{conv}} \left( \int_{I_i} \left( \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{\text{conv}} k(z, I_j, V([z_j, z_{j+1}])) \right) \right). \]  
(3.21)
Using (3.15), (3.16), and the properties of measure of noncompactness $\alpha$, we have

$$
\alpha(F(V(t))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \alpha \left( f \left( z, V(I_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j) \text{conv} k(z, I_j, V([z_j, z_{j+1}])) \right) \right)
$$

$$
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d_1 \cdot \max \left\{ \alpha(V(I_i)), \alpha \left( \sum_{j=0}^{m-1} (z_{j+1} - z_j) \text{conv} k(z, I_j, V([z_j, z_{j+1}])) \right) \right\}.
$$

Let us observe that

(iii) if $\alpha(V(I_i)) > \alpha(\sum_{j=0}^{m-1} (z_{j+1} - z_j) \text{conv} k(z, I_j, V([z_j, z_{j+1}])))$, then

$$
\alpha(V) = \alpha(\text{conv}(\{x \cup G(V)\})) \leq \alpha(F(V)) < d_1 \cdot c \cdot \alpha(V),
$$

because $d_1 \cdot c < 1$ so $\alpha(V) < \alpha(V)$, a contradiction;

(iv) if $\alpha(V(I_i)) < \alpha(\sum_{j=0}^{m-1} (z_{j+1} - z_j) \text{conv} k(z, I_j, V([z_j, z_{j+1}])))$, then

$$
\alpha(V) < \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d_1 \cdot \alpha \left( \sum_{j=0}^{m-1} (z_{j+1} - z_j) \text{conv} k(z, I_j, V([z_j, z_{j+1}])) \right)
$$

$$
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d_1 \cdot \sum_{j=0}^{m-1} (z_{j+1} - z_j) \alpha(k(z, I_j, V([z_j, z_{j+1}])))
$$

$$
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d_1 \cdot \sum_{j=0}^{m-1} (z_{j+1} - z_j) \sup_{s \in I_j} d_2(z, s) \alpha(V([z_j, z_{j+1}]))
$$

$$
= d_1 \cdot c \cdot \sum_{j=0}^{m-1} (z_{j+1} - z_j) \cdot d_2(z, p_j) v(s_j)
$$

$$
= d_1 \cdot c \left[ \sum_{j=0}^{m-1} (z_{j+1} - z_j) \cdot d_2(z, p_j) v(p_j)
$$

$$
+ \sum_{j=0}^{m-1} (z_{j+1} - z_j) \cdot d_2(z, p_j) (v(s_j) - v(p_j)) \right].
$$

By continuity of $v$ we have $v(s_j) - v(p_j) < \epsilon$ and $\epsilon \to 0$ if $m \to \infty$, so

$$
v(t) = \alpha(V(t)) \leq d_1 \cdot c \cdot \sup_{z \in I_c} \int_0^c d_2(z, s) v(s) ds.
$$

By (3.17) we have $v(t) = \alpha(V(t)) = 0$ for $t \in I_c$.

Using Arzelá-Ascoli theorem, we obtain that $V$ is relatively compact. By Theorem 1.6 the operator $F$ has a fixed point. This means that there exists a solution of the problem (1.1).
References


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