The aim of this work is to describe the orthogonal polynomials sequences which are identical to their second associated sequence. The resulting polynomials are semiclassical of class $s \leq 3$. The characteristic elements of the structure relation and of the second-order differential equation are given explicitly. Integral representations of the corresponding forms are also given. A striking particular case is the case of the so-called electrospheric polynomials.

1. Introduction

A long time ago [4], Guille and Aubert wrote a paper on electrospheric polynomials. They are a particular case of orthogonal polynomials which are identical to their second associated sequence. This property has not been noticed. More recently [7], the first author studied the second-order self-associated sequences in the case where they are positive definite.

Here, we will describe all the orthogonal sequences which are identical to their second associated sequence. Such a sequence depends on three parameters $(\tau, \upsilon, \varepsilon)$, where $\tau \in \mathbb{C}$, $\upsilon \in \mathbb{C} - \{-1, 1\}$, and $\varepsilon^2 = 1$.

When $\tau = 0$, we obtain the so-called electrospheric polynomials. When $|\tau| \leq \min(1, |\upsilon|)$, we have the positive definite case.

In Section 2, we deal with general features. Section 3 is devoted to the classification of second-order self-associated sequences. In Section 4, we carry out the quadratic decomposition of second-order self-associated sequences. This section is necessary for determining the useful materials needed in Section 5 in which we establish the structure relation between any second-order self-associated sequence and the differential equation fulfilled by any polynomial of such a sequence. Finally, in Section 6, we give the integral representation and the moments of the corresponding forms.

2. Preliminary results

2.1. Computing forms and Stieltjes function. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$.
on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of $u$. For any form $u$ and any polynomial $h$, we let $Du = u'$ and $hu$ be the forms defined by duality:

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle hu, f \rangle := \langle u, hf \rangle, \quad f \in \mathcal{P}. \quad (2.1)$$

We recall the definition of right multiplication of a form by a polynomial:

$$(up)(x) := \left\langle u, \frac{xp(x) - \xi p(\xi)}{x - \xi} \right\rangle, \quad u \in \mathcal{P}', \; p \in \mathcal{P}. \quad (2.2)$$

By duality, we obtain the Cauchy’s product of two forms:

$$\langle uv, p \rangle := \langle u, vp \rangle, \quad u, v \in \mathcal{P}', \; p \in \mathcal{P}. \quad (2.3)$$

We define [1] the form $(x - c)^{-1}u$, $c \in \mathbb{C}$, through

$$\langle (x - c)^{-1}u, p \rangle := \langle u, \theta_c p \rangle, \quad (2.4)$$

with

$$(\theta_c p)(x) := \frac{p(x) - p(c)}{x - c}, \quad u \in \mathcal{P}', \; p \in \mathcal{P}. \quad (2.5)$$

From the definitions, we have $(u\theta_0 f)(x) = \langle u, (f(x) - f(\xi))/(x - \xi) \rangle$, $u \in \mathcal{P}'$, $f \in \mathcal{P}$.

Hence, $W_n^{(1)}(x) = (w_0 \theta_0 W_{n+1})(x)$.

We introduce the operator $\sigma : \mathcal{P} \to \mathcal{P}$ defined by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$.

By transposition, we define $\sigma u$ by duality:

$$\langle \sigma u, f \rangle = \langle u, \sigma f \rangle, \quad \forall u \in \mathcal{P}', \; \forall f \in \mathcal{P}. \quad (2.6)$$

Consequently, $(\sigma u)_n = (u)_{2n}$. The following results are fundamental [1, 13].

**Lemma 2.1.** For any $f, g \in \mathcal{P}$, $u, v \in \mathcal{P}'$, and $c \in \mathbb{C}$,

$$f(x)(uv) = (f(x)v)u + x(v\theta_0 f)(x)u, \quad (2.7)$$

$$(x - c)^{-1}(fu) = f(c)((x - c)^{-1}u) + (\theta_c f)u - \langle u, \theta_c f \rangle \delta_c \quad (\langle \delta_c, f \rangle = f(c)), \quad (2.8)$$

$$f((x - c)^{-1}u) = f(c)((x - c)^{-1}u) + (\theta_c f)u, \quad (2.9)$$

$$fu' = fu' + f'u, \quad (2.10)$$

$$(u\theta_0 f)(x) = (\theta_0 uf)(x), \quad (2.11)$$

$$f(x)\sigma u = \sigma(f(x^2)u), \quad (2.12)$$

$$2(\sigma u)' = \sigma((x^{-1}u)'), \quad (2.13)$$

$$\sigma u' = 2(\sigma xu)', \quad (2.14)$$
We will also use the so-called formal Stieltjes function associated with \( u \in \mathcal{P}' \) and defined by

\[
S(u)(z) := -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}}.
\] (2.15)

**Lemma 2.2.** For any \( f \in \mathcal{P} \) and \( u, v \in \mathcal{P}' \),

\[
S(fu)(z) = f(z)S(u)(z) + (u\theta_0 f)(z),
\]

\[
S(u')(z) = S'(u)(z),
\]

\[
S(uv)(z) = -zS(u)(z)S(v)(z),
\]

\[
S(u^n)(z) = (-1)^{n-1}z^{n-1}(S(u)(z))^n, \quad n \geq 1,
\]

\[
S(x^{-n}u)(z) = z^{-n}S(u)(z), \quad n \geq 1.
\] (2.16)

### 2.2. Dual sequences and orthogonal sequences

Let \( \{W_n\}_{n \geq 0} \) be a monic polynomials sequence (MPS), \( \deg W_n = n, n \geq 0 \), and let \( \{w_n\}_{n \geq 0} \) be its dual sequence, \( w_n \in \mathcal{P}' \), defined by

\[
\langle w_n, W_m \rangle := \delta_{n,m}, \quad n, m \geq 0.
\] (2.19)

The sequence \( \{W_n\}_{n \geq 0} \) is orthogonal with respect to \( w \); it is a monic orthogonal polynomials sequence (MOPS). Necessarily, \( w = \lambda w_0, \lambda \neq 0 \). In this case, we have \( w_n = (\langle w_0, W_n^2 \rangle)^{-1}W_n w_0, n \geq 0 \), and \( \{W_n\}_{n \geq 0} \) fulfils the following second-order recurrence relation:

\[
W_0(x) = 1, \quad W_1(x) = x - \beta_0,
\]

\[
W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - y_{n+1}W_n(x), \quad n \geq 0.
\] (2.20)
Likewise, the sequence \( \{W_n^{(1)}\}_{n \geq 0} \) verifies the recurrence relation
\[
W_0^{(1)}(x) = 1, \quad W_1^{(1)}(x) = x - \beta_1,
\]
\[
W_{n+2}(x) = (x - \beta_{n+2}) W_{n+1}^{(1)}(x) - \gamma_{n+2} W_n^{(1)}(x), \quad n \geq 0,
\] (2.21)
and it is orthogonal with respect to \( w_0^{(1)} \), where \( \gamma_1 w_0^{(1)} = -x^2 w_0^{-1} \). (2.22)

Through the formal Stieltjes function \[16\],
\[
\gamma_1 S(w_0^{(1)})(z) = -\frac{1}{S(w_0)(z)} - (z - \beta_0).
\] (2.23)

The successive associated sequences are defined recursively:
\[
W_n^{(r+1)} = (W_n^{(r)})^{(1)}, \quad w_n^{(r+1)} = (w_n^{(r)})^{(1)}, \quad n, r \geq 0.
\] (2.24)

The sequence \( \{W_n^{(r+1)}\}_{n \geq 0} \) satisfies the recurrence relation
\[
W_0^{(r+1)}(x) = 1, \quad W_1^{(r+1)}(x) = x - \beta_{r+1},
\]
\[
W_{n+2}(x) = (x - \beta_{n+r+2}) W_{n+1}^{(r+1)}(x) - \gamma_{n+r+2} W_n^{(r+1)}(x), \quad n \geq 0.
\] (2.25)

From (2.23), we have
\[
\gamma_{n+r+1} S(w_0^{(n+r+1)})(z) = -\frac{1}{S(w_0^{(n+r)})(z)} - (z - \beta_{n+r}), \quad n, r \geq 0.
\] (2.26)

Hence, we get \[6, 10, 13\]
\[
\gamma_{n+r+1} S(w_0^{(n+r+1)})(z) = -\frac{W_n^{(r+1)}(z) + W_{n+1}^{(r)}(z) S(w_0^{(r)})(z)}{W_{n-1}^{(r+1)}(z) + W_n^{(r)}(z) S(w_0^{(r)})(z)}, \quad n, r \geq 0.
\] (2.27)

Let \( \{W_n\}_{n \geq 0} \) be an MPS. It is always possible to associate with it two MPSs \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \), \( \deg P_n = \deg R_n = n, \quad n \geq 0 \), and two polynomials sequences \( \{a_n(x)\}_{n \geq 0} \) and \( \{b_n(x)\}_{n \geq 0} \) such that
\[
W_{2n}(x) = P_n(x^2) + x a_{n-1}(x^2),
\]
\[
W_{2n+1}(x) = x R_n(x^2) + b_n(x^2), \quad n \geq 0,
\] (2.28)
where \( \deg a_n \leq n \) and \( \deg b_n \leq n \).
Since \( \deg P_n = \deg R_n = n, n \geq 0 \), there exist two tables of coefficients \((\lambda^n_\nu)\) and \((\theta^n_\nu)\), \(0 \leq \nu \leq n, n \geq 0\), such that

\[
a_n(x) = \sum_{\nu=0}^{n} \lambda^n_\nu R_n(x), \quad n \geq 0,
\]
\[
b_n(x) = \sum_{\nu=0}^{n} \theta^n_\nu P_n(x), \quad n \geq 0.
\]

(2.29)

2.3. Semiclassical forms. Let \( \Phi \) (monic) and \( \Psi \) be two polynomials \( (\deg \Psi = p \geq 1, \deg \Phi = t) \). A form \( w \) is called semiclassical when it is regular and satisfies the equation \([8, 11]\)

\[
(\Phi w)' + \Psi w = 0.
\]

(2.30)

When \( w \) is semiclassical, the orthogonal sequence \( \{W_n\}_{n \geq 0} \) is also called semiclassical.

The pair \((\Phi, \Psi)\) is not unique. Equation (2.30) can be simplified if and only if there exists a root \( c \) of \( \Phi \) such that

\[
\Psi(c) + \Phi'(c) = 0, \quad \langle w, \theta_c \Psi + \theta^2_c \Phi \rangle = 0.
\]

(2.31)

Then \( u \) fulfills the equation \(((\theta_c \Phi)w)' + \{\theta_c \Psi + \theta^2_c \Phi\}w = 0\).

We call the class of \( w \) the minimum value of the integer \( \max(\deg \Phi - 2, \deg \Psi - 1) \) for all pairs satisfying (2.30). Given the pair \((\Phi_0, \Psi_0)\), the class \( s \geq 0 \) is unique. When \( s = 0 \), the form \( w \) is classical (Hermite, Laguerre, Bessel, Jacobi).

When the form \( w \) is of class \( s \), the orthogonal sequence associated with respect to \( w \) is known to be of class \( s \).

The class of semiclassical forms is \( s \) if and only if the following condition is satisfied \([11]\):

\[
\prod_{c \in \Theta} (|\Psi(c) + \Phi'(c)| + |\langle w, \theta_c \Psi + \theta^2_c \Phi \rangle|) \neq 0,
\]

(2.32)

where \( \Theta = \{c, \phi(c) = 0\} \).

Lemma 2.3. Let \( w \) be a regular semiclassical form verifying (2.30). Let \( a \) be a root of \( \Phi \) such that

\[
|\Psi(a) + \Phi'(a)| + |\langle w, \theta_a \Psi + \theta^2_a \Phi \rangle| = 0,
\]

(3.33)

\[
|\Psi(c) + \Phi'(c)| + |\langle w, \theta_c \Psi + \theta^2_c \Phi \rangle| \neq 0,
\]

(3.34)

for all \( c \) roots of \( \Phi \) different from \( a \). Then the form \( w \) satisfies the equation

\[
(\Phi_1 w)' + \Psi_1 w = 0.
\]

(2.35)
where $\Phi_1 = \theta_a \Phi$ and $\Psi_1 = \theta_a \Psi + \theta_a^2 \Phi$ such that

$$|\Psi_1(c) + \Phi_1'(c)| + |\langle w, \theta_c \Psi_1 + \theta_c^2 \Phi_1 \rangle| \neq 0$$

(2.36)

for all $c$ roots of $\Phi$ different from $a$.

Proof. We suppose that there exists a root $c$ of $\Phi$ different from $a$ verifying

$$\Psi_1(c) + \Phi_1'(c) = 0, \quad \langle w, \theta_c \Psi_1 + \theta_c^2 \Phi_1 \rangle = 0.$$  

(2.37)

We have

$$\Phi(x) = (x - a) \Phi_1(x), \quad (\Psi + \Phi_1)(x) = (x - a) \Psi_1(x);$$  

(2.38)

then

$$\Psi(c) + \Phi'(c) = (c - a)(\Psi_1(c) + \Phi_1'(c)), \quad \theta_c \Psi + \theta_c^2 \Phi = \Psi_1 - (c - a)(\theta_c \Psi_1 + \theta_c^2 \Phi_1).$$  

(2.39)

On account of $\langle w, \Psi_1 \rangle = 0$, we deduce that $\Psi(c) + \Phi'(c) = 0$ and $\langle w, \theta_c \Psi + \theta_c^2 \Phi \rangle = 0$. This contradicts the conditions given in (2.34). \hfill \square

2.4. Affine transformation. We define the linear operators $\tau_b$ and $h_a$ in $\mathcal{P}'$ as follows:

\[
\langle \tau_b u, p \rangle = \langle u, \tau_{-b} p \rangle = \langle u, p(x + b) \rangle, \quad b \in \mathbb{C}, \quad u \in \mathcal{P}', \quad p \in \mathcal{P},
\]

\[
\langle h_a u, p \rangle = \langle u, h_a p \rangle = \langle u, p(ax) \rangle, \quad a \in \mathbb{C} - \{0\}, \quad u \in \mathcal{P}', \quad p \in \mathcal{P}. \tag{2.40}
\]

Let $\{W_n\}_{n \geq 0}$ be an MPS with its dual sequence $\{w_n\}_{n \geq 0}$. The dual sequence $\{\hat{w}_n\}_{n \geq 0}$ of $\{W_n\}_{n \geq 0}$ with $\hat{w}_n(x) = a^{-n} W_n(ax + b), n \geq 0, a \neq 0$, is given by $\hat{w}_n = a^n(h_a^{-1} \circ \tau_{-b})w_n$, $n \geq 0$.

Let $\{W_n\}_{n \geq 0}$ be an MOPS with respect to $w$. Then $\{\hat{W}_n\}_{n \geq 0}$ is an MOPS with respect to $\hat{w} = (h_a^{-1} \circ \tau_{-b})w$. We have

\[
\tilde{\beta}_n = \frac{\beta_n - b}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \tag{2.41}
\]

Lemma 2.4. For any $f \in \mathcal{P}, u, v \in \mathcal{P}'$, and $(a, b) \in \mathbb{C} - \{0\} \times \mathbb{C}$, [8, 13],

\[
\tau_b(fu) = (\tau_b f)(\tau_b u), \tag{2.42}
\]

\[
h_a(fu) = (h_a^{-1} f)(h_a u), \tag{2.43}
\]

\[
\tau_b(uv) = (\tau_b u)(\tau_b v)\delta_b^{-1}, \tag{2.44}
\]

\[
h_a(uv) = (h_a u)(h_a v). \tag{2.45}
\]
As a result, if \( w \) is a semiclassical form of class \( s \) satisfying (2.30), then the shifted form \( \tilde{w} = (h_{a^{-1}} \circ \tau_b)w \) is of class \( s \) satisfying the equation

\[
(\tilde{\Phi} \tilde{w})' + \tilde{\Psi} \tilde{w} = 0,
\]

where

\[
\tilde{\Phi}(x) = a^{-1}\Phi(ax + b), \quad \tilde{\Psi}(x) = a^{1-i}\Psi(ax + b).
\]

(2.47)

**Lemma 2.5.** Let \( \{W_n\}_{n \geq 0} \) be an MPS, \( \deg W_n = n, n \geq 0 \), and let \( \{w_n\}_{n \geq 0} \) be its dual sequence. For any \((a, b) \in \mathbb{C} - \{0\} \times \mathbb{C}\),

\[
\tau_b (w_n^{(1)}) = (\tau_b w_n)^{(1)},
\]

(2.48)

\[
h_a (w_n^{(1)}) = (h_a w_n)^{(1)}.
\]

(2.49)

*Proof.* By multiplying the two sides of (2.18) by the form \( w_0 \), we obtain

\[
w_n^{(1)}w_0 = xw_{n+1}.
\]

(2.50)

By introducing the operator \( \tau_b \) in the last expression, from (2.42) and (2.44), we obtain

\[
(\tau_b (w_n^{(1)}))(\tau_b w_0) = ((x - b)(\tau_b w_{n+1}))\delta_b.
\]

(2.51)

From (2.7),

\[
(\tau_b (w_n^{(1)}))(\tau_b w_0) = ((x - b)\delta_b)(\tau_b w_{n+1}) + x(\tau_b w_{n+1})
\]

\[
- x((\tau_b w_{n+1})\theta_0(\xi - b))(x)\delta_b.
\]

(2.52)

Since

\[
(x - b)\delta_b = 0, \quad ((\tau_b w_{n+1})\theta_0(\xi - b))(x) = 0, \quad n \geq 0,
\]

(2.53)

then

\[
(\tau_b (w_n^{(1)}))(\tau_b w_0) = x(\tau_b w_{n+1}), \quad n \geq 0,
\]

(2.54)

or

\[
\tau_b (w_n^{(1)}) = (x(\tau_b w_{n+1}))(\tau_b w_0)^{-1}, \quad n \geq 0.
\]

(2.55)

From (2.18) and (2.55), we deduce (2.48).
To prove (2.48), we introduce the operator $h_a$ in the expression (2.50). From (2.43) and (2.45), we give
\[(h_a(w_n^{(1)}))(h_aw_0) = a^{-1}x(h_aw_{n+1}), \quad n \geq 0.\] (2.56)

But
\[\left(a^{-n}h_aw_n\right)^{(1)} = x(a^{-(n+1)}h_aw_{n+1})(h_aw_0)^{-1}, \quad n \geq 0.\] (2.57)

From (2.18) and (2.57), we deduce (2.49). □

2.5. Second-degree forms. The form $w$ is a second-degree form [13] if it is regular and if there exist polynomials $B$ and $C$ such that
\[B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \quad (2.58)\]

where $D$ depends on $B$, $C$, and $w$.

The regularity of $w$ means that we must have
\[B \neq 0, \quad C^2 - 4BD \neq 0, \quad D \neq 0. \quad (2.59)\]

The following expressions are equivalent to (2.58), [13]:
\[B(x)w^2 = xC(x)w, \quad \langle w^2, \theta_0B \rangle = (w, C). \quad (2.60)\]

In the sequel, we will assume $B$ to be monic and we will be looking for any regular form $w$ verifying $(w)_0 = 1$.

A second-degree form $w$ is a semiclassical form and satisfies (2.30), where [13]
\[k\phi(x) = B(x)(C^2(x) - 4B(x)D(x)), \quad \phi \text{ monic, } k \neq 0, \quad (2.61)\]
\[k\psi(x) = -\frac{3}{2}B(x)(C^2(x) - 4B(x)D(x))'.\]

3. The second-order self-associated orthogonal sequences and their classification

In this section, we quote the second-order self-associated sequences following the class of their corresponding canonical forms.

Definition 3.1. Let any integer $m \geq 1$ be fixed. Then the MOPS $\{W_n\}_{n \geq 0}$ is called an $m$-order self-associated polynomials sequence when it fulfils
\[W_n^{(m)} = W_n, \quad n \geq 0. \quad (3.1)\]

In this case, the form $w_0$ is also called an $m$-order self-associated form. See also [14, 15].
Then \( w_0 \) satisfies
\[
 w_0^{(m)} = w_0. \tag{3.2}
\]

From (3.1), the coefficients of (2.20) are given by
\[
 \beta_{n+m} = \beta_n, \quad \gamma_{n+m+1} = \gamma_{n+1}, \quad n \geq 0. \tag{3.3}
\]

The case \( m = 1 \) is well known; \( w_0 \) is the Tchebychev form of the second kind.

According to Lemma 2.5, we give the following result.

**Proposition 3.2.** Let \( \{W_n\}_{n \geq 0} \) be an \( m \)-order self-associated MPS, \( \deg W_n = n, n \geq 0, \) and let \( \{w_n\}_{n \geq 0} \) be its dual sequence. Then the shifted sequence form \( \{\~w_n\}_{n \geq 0} \) fulfills
\[
 \~w_n^{(m)} = \~w_n, \quad m \in \mathbb{N} - \{0\}, \quad n \geq 0, \tag{3.4}
\]
where
\[
 \~w_n = a^n (h_{a+1} \circ \tau_{-b}) w_n, \quad b \in \mathbb{C}, \quad a \in \mathbb{C} - \{0\}, \quad n \geq 0. \tag{3.5}
\]

The object of this subject is to treat the case where \( m = 2 \) by describing all the second-order self-associated polynomials sequences and their classification. We denote by \( \{Z_n\}_{n \geq 0} \) these polynomials sequences and \( \{z_n\}_{n \geq 0} \) their dual sequences. From (3.3), we get
\[
 \beta_{n+2} = \beta_n, \quad \gamma_{n+3} = \gamma_{n+1}, \quad n \geq 0. \tag{3.6}
\]

This implies
\[
 \begin{align*}
 \beta_{2n} &= \beta_0, \quad \beta_{2n+1} = \beta_1, \quad n \geq 0, \\
 \gamma_{2n+1} &= \gamma_1, \quad \gamma_{2n+2} = \gamma_2, \quad n \geq 0. \tag{3.7}
\end{align*}
\]

For \( \alpha = (1/2)(\beta_0 + \beta_1), \beta = (1/2)(\beta_0 - \beta_1), \lambda = (1/2)(\gamma_2 + \gamma_1), \mu = (1/2)(\gamma_1 - \gamma_2), n \geq 0, \) we have
\[
 \begin{align*}
 \beta_n &= \alpha + (-1)^n \beta, \quad n \geq 0, \quad (\alpha, \beta) \in \mathbb{C}^2, \\
 \gamma_{n+1} &= \lambda + (-1)^n \mu, \quad n \geq 0, \quad (\lambda, \mu) \in \mathbb{C}^2, \lambda^2 \neq \mu^2. \tag{3.8}
\end{align*}
\]

By means of (2.23), we have
\[
 \begin{align*}
 \gamma_2 S(z_0^{(2)})(z) &= \frac{1}{S(z_0^{(1)})(z)} - (z - \beta_1), \tag{3.9} \\
 \gamma_1 S(z_0^{(1)})(z) &= \frac{1}{S(z_0)(z)} d - (z - \beta_0). \tag{3.10}
\end{align*}
\]
Substituting (3.10) into (3.9), we obtain
\[ γ_2 S(z_0^{(2)}) (z) = \frac{γ_1 S(z_0)(z)}{1 + (z - β_0) S(z_0)(z)} - (z - β_1). \] (3.11)

Since
\[ z_0^{(2)} = z_0, \] (3.12)
relation (3.11) becomes
\[ (z - β_0) S^2(z_0)(z) + \frac{1}{γ_2} (γ_2 - γ_1 + (z - β_0) (z - β_1)) S(z_0)(z) + \frac{1}{γ_2} (z - β_1) = 0. \] (3.13)

From (3.8), we get
\[ (z - α - β) S^2(z_0)(z) + \frac{1}{λ - μ} (z^2 - 2αz + α^2 - β^2 - 2μ) S(z_0)(z) + \frac{1}{λ - μ} (z - α + β) = 0. \] (3.14)

Thus, the form \( z_0 \) is a second-degree form [10, 14, 15]. It is also a semiclassical form of class \( s ≤ 3 \), satisfying the functional equation (2.30) with
\[ Φ(x) = (x - (α + β))( (x - α)^2 - 2λ - β^2)^2 - 4(λ^2 - μ^2) \),
\[ Ψ(x) = -6(x - α)(x - (α + β))( (x - α)^2 - 2λ - β^2) \]. (3.15)

Let \( δ_1, δ_2 \) be two complex numbers such that
\[ δ_1^2 = 2λ + β^2 + 2\sqrt{λ^2 - μ^2}, \quad δ_2^2 = 2λ + β^2 - 2\sqrt{λ^2 - μ^2}. \] (3.16)

The polynomial \( Φ \) becomes
\[ Φ(x) = (x - α - β)(x - α - δ_1)(x - α + δ_1)(x - α - δ_2)(x - α + δ_2). \] (3.17)

We remark that \( δ_1^2 - δ_2^2 = 4\sqrt{λ^2 - μ^2} \). The regularity of \( z_0 \) leads to \( λ^2 ≠ μ^2 \). Then \( δ_1^2 ≠ δ_2^2 \); so necessarily one of these values is different from zero. We can suppose that \( δ_1 ≠ 0 \).

We make a suitable shift such that \( α = 0 \) and \( δ_1 = 1 \). With \( β = τ \) and \( δ_2 = υ \), from (3.16), we have \( λ = (1/4)(1 - 2τ^2 + υ^2) \) and \( μ = (1/2)eτ, ε = ±1 \), where
\[ ζτ,υ = \sqrt{(τ^2 - 1)(τ^2 - υ^2)}. \] (3.18)

Therefore, (3.14) becomes
\[ (z - τ) S^2(z_0)(z) + \frac{1}{γ_2} (z^2 - τ^2 - εζτ,υ) S(z_0)(z) + \frac{1}{γ_2} (z + τ) = 0, \] (3.19)
where

\[ \gamma_2 = \frac{1}{4} (1 - 2\tau^2 + \nu^2 - 2\varepsilon \xi \tau, \nu). \]  

(3.20)

The functional equation fulfilled by the form \( z_0 \) becomes

\[ (\Phi z_0)' + \Psi z_0 = 0, \]  

(3.21)

where

\[ \Phi(x) = (x - \tau)(x^2 - 1)(x^2 - \nu^2), \]  

(3.22)

\[ \Psi(x) = -3x(x - \tau)(2x^2 - 1 - \nu^2). \]  

(3.23)

**Proposition 3.3.** Let \( \{Z_n\}_{n \geq 0} \) be a second-order self-associated polynomials sequence with respect to \( z_0 \). Then there exists \( (\tau, \nu) \in \mathbb{C}^2, \nu^2 \neq 1 \), such that

\[ Z_0(x) = 1, \quad Z_1(x) = x - \tau, \]  

\[ Z_{n+2}(x) = (x - (-1)^{n+1} \tau)Z_{n+1}(x) - \left( \frac{1}{4} (1 - 2\tau^2 + \nu^2) + \frac{(-1)^n}{2} \varepsilon \xi \tau, \nu \right)Z_n(x), \quad n \geq 0. \]  

(3.24)

The form \( z_0 \) is a semiclassical form of class \( s \leq 3 \) and satisfies the functional equation (3.21), with the following initial conditions:

\[ \langle z_0, 1 \rangle = 1, \quad \langle z_0, x \rangle = \tau, \quad \langle z_0, x^2 \rangle = \frac{1}{4} (1 + 2\tau^2 + \nu^2) + \frac{1}{2} \varepsilon \xi \tau, \nu, \]  

\[ \langle z_0, x^3 \rangle = \tau \langle z_0, x^2 \rangle. \]  

(3.25)

Noting that the sequence \( \{Z_n^{(1)}\}_{n \geq 0} \) is also a second-order self-associated sequence,

\[ (Z_n(\tau, \nu, \varepsilon; x))^{(1)} = Z_n(-\tau, \nu, -\varepsilon; x), \quad n \geq 0. \]  

(3.26)

**Proof.** Let \( \{W_n\}_{n \geq 0} \) be an MOPS satisfying (2.20) with respect to \( w_0 \). Generally, we have

\[ \langle w_0, x \rangle = \beta_0, \quad \langle w_0, x^2 \rangle = \beta_0^2 + \gamma_1, \quad \langle w_0, x^3 \rangle = \beta_0^3 + 2\beta_0 \gamma_1 + \beta_1 \gamma_1. \]  

(3.27)

By means of relations (3.8), (3.22), and (3.23), we deduce the result. \( \square \)

In the sequel, we quote all the second-order self-associated MPSs \( \{Z_n\}_{n \geq 0} \). For this, we need the following lemma. Let \( c \) be a root of \( \Phi \). We have \( c \in \{-1, 1, \tau, -\nu, \nu\} \).

**Lemma 3.4.** Let \( \{Z_n\}_{n \geq 0} \) be a second-order self-associated polynomials sequence with respect to \( z_0 \). The expressions \( \Phi'(c) + \Psi(c) \) and \( \langle z_0, \theta_c^2 \Phi + \theta_c \Psi \rangle \) are given for all \( c \) roots of \( \Phi \) in **Table 3.1**.

**Proof.** From (3.22) and (3.23), a simple calculation gives us the values of \( \Phi'(c) + \Psi(c) \) for all \( c \) roots of \( \Phi \).
From the expressions of the moments \((3.28)\), we deduce the results of Table 3.1.

<table>
<thead>
<tr>
<th>(\Phi' \Phi)</th>
<th>(\Phi + \Psi)</th>
<th>(\langle z_0, \theta^2 \Phi + \Psi \rangle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((\tau - 1)(1 - v^2))</td>
<td>(2(\tau^2 - 1 - \xi_{\tau,v}))</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-((\tau + 1)(1 - v^2)))</td>
<td>(-2(\tau^2 - 1 - \xi_{\tau,v}))</td>
</tr>
<tr>
<td>(v)</td>
<td>(-v(v - \tau)(v^2 - 1))</td>
<td>(2v(\tau^2 - v^2 - \xi_{\tau,v}))</td>
</tr>
<tr>
<td>(-v)</td>
<td>(-v(v + \tau)(v^2 - 1))</td>
<td>(-2v(\tau^2 - v^2 - \xi_{\tau,v}))</td>
</tr>
<tr>
<td>(\tau)</td>
<td>((\tau^2 - 1)(\tau^2 - v^2))</td>
<td>(-2\tau \xi_{\tau}(\tau^2 - 1)(\tau^2 - v^2))</td>
</tr>
</tbody>
</table>

For calculating \(\langle z_0, \theta^2 \Phi + \Psi \rangle\), we must initially calculate the polynomials \(\theta^2 \Phi + \Psi\)\((x)\) explicitly. Through definition (3.1) and (3.22), (3.23), we have

\[
(\theta_1^2 \Phi + \theta_1 \Psi)(x) = -5x^3 + (5\tau - 4)x^2 + (2v^2 + 4\tau - 1)x + v^2 - 2v^2\tau + \tau - 1,
\]
\[
(\theta_2^2 \Phi + \theta_2 \Psi)(x) = -5x^3 + (5\tau + 4)x^2 + (2v^2 - 4\tau - 1)x - v^2 - 2v^2\tau + \tau + 1,
\]
\[
(\theta_3^2 \Phi + \theta_3 \Psi)(x) = -5x^3 + \tau x^2 + (2v^2 + \tau^2 + 2)x + \tau x^2 - \tau v^2 - \tau,
\]
\[
(\theta_4^2 \Phi + \theta_4 \Psi)(x) = -5x^3 + (5\tau + 4v)x^2 + (4\tau v - v^2 + 2)x + \tau v^2 - v^3 + v - 2\tau,
\]
\[
(\theta_5^2 \Phi + \theta_5 \Psi)(x) = -5x^3 + (5\tau + 4v)x^2 + (-4\tau v - v^2 + 2)x + \tau v^2 + v^3 - v - 2\tau.
\]

From the expressions of the moments \((z_0)_k\), \(0 \leq k \leq 3\), given by (3.25), and relations (3.28), we deduce the results of Table 3.1. \(\square\)

**Proposition 3.5.** Let \(\{Z_n\}_{n \geq 0}\) be a second-order self-associated MPS with respect to \(z_0\) (remember that the regularity of \(z_0\) means \(v^2 \neq 1\)). Denoting by \(s\) the class of \(z_0\),

(a) if \(\tau^2 \neq 1, \tau^2 \neq v^2\), and \(v \neq 0\), so \(s = 3\) and \(z_0\) is given by (3.21), (3.22), (3.23), (3.24), and (3.25);

(b) if \(v \neq 0\) and \(\tau = 1\), so \(s = 2\) and \(z_0\) is given by

\[
((x^2 - 1)(x^2 - v^2)(z_0)') + (5x^3 + x^2 + (3 + 2v^2)x - v^2)z_0 = 0,
\]

where

\[
(z_0)_1 = 1, \quad (z_0)_2 = \frac{1}{4} (v^2 + 3),
\]

and

\[
\beta_n = (-1)^n, \quad \gamma_{n+1} = \frac{v^2 - 1}{4}, \quad v^2 \neq 1, v \neq 0, n \geq 0;
\]

(c) if \(v = 0, \tau^2 \neq 1, \text{ and } \tau \neq 0\), so \(s = 2\) and \(z_0\) is given by

\[
(x(x - \tau)(x^2 - 1)(z_0)' + (x - \tau)(-5x^2 + 2)z_0 = 0,
\]
where
\[
(z_0)_1 = \tau, \quad (z_0)_2 = \frac{1}{4}(1 + 2\tau^2) + \frac{1}{2}\varepsilon\tau\sqrt{\tau^2 - 1},
\] (3.33)

and
\[
\beta_n = (-1)^n\tau, \quad \gamma_{n+1} = -\frac{1}{4}(\tau - (-1)^n\varepsilon\sqrt{\tau^2 - 1})^2, \quad \tau^2 \neq 1, \tau \neq 0, n \geq 0;
\] (3.34)

(d) if \( v = 0 \) and \( \tau = 1 \), so \( s = 1 \) and \( z_0 \) is given by
\[
(x(x^2 - 1)z_0)′ + (-4x^2 + x + 2)z_0 = 0, \quad (z_0)_1 = 1,
\] (3.35)
\[
\beta_n = (-1)^n, \quad \gamma_{n+1} = -\frac{1}{4}, \quad n \geq 0;
\]

(e) if \( v = 0 \) and \( \tau = 0 \), so \( s = 0 \) and \( z_0 \) is the Tchebychev form of the second kind \([10, 12, 13]\), given by
\[
((x^2 - 1)z_0)′ - 3xz_0 = 0,
\] (3.36)
\[
\beta_n = 0, \quad \gamma_{n+1} = \frac{1}{4}, \quad n \geq 0.
\] (3.37)

Proof. (a) In the case \( \tau^2 \neq 1, \tau^2 \neq \upsilon^2 \), and \( \upsilon \neq 0 \) and from Table 3.1, we have
\[
|\Psi(c) + \Phi′(c)| + |\langle z_0, \theta_c\Psi + \theta_c^2\Phi \rangle| \neq 0
\] (3.38)
for all \( c \) roots of \( \Phi \). Relation (2.32) is realized. Consequently, (3.21) is not simplified, so the form \( z_0 \) is of class \( s = 3 \).

(b) In the second case, the functional equation of \( z_0 \) is given by
\[
((x - 1)(x^2 - 1)(x^2 - \upsilon^2)z_0)′ - 3x(x - 1)(2x^2 - 1 - \upsilon^2)z_0 = 0.
\] (3.39)
From Table 3.1, \( \Psi(1) + \Phi′(1) = 0, \langle z_0, \theta_1\Psi + \theta_1^2\Phi \rangle = 0 \), and \( |\Psi(c) + \Phi′(c)| + |\langle z_0, \theta_c\Psi + \theta_c^2\Phi \rangle| \neq 0 \) for all \( c \in \{-1, \upsilon, -\upsilon\} \).

Then this equation is simplified by \( x - 1 \), and \( z_0 \) fulfills
\[
(\Phi_1z_0)′ + \Psi_1z_0 = 0,
\] (3.40)
where \( \Phi_1(x) = (x^2 - 1)(x^2 - \upsilon^2) \) and \( \Psi_1(x) = -5x^3 + x^2 + (3 + 2\upsilon^2)x - \upsilon^2 \).

From Lemma 2.3,
\[
|\Psi_1(c) + \Phi_1′(c)| + |\langle z_0, \theta_c\Psi_1 + \theta_c^2\Phi_1 \rangle| \neq 0
\] (3.41)
for all $c \in \{-1, \upsilon, -\upsilon\}$; and taking into account $\Psi_1'(1) + \Phi'_1(1) = (1 - \upsilon^2) \neq 0$, we deduce the result.

When $\upsilon \neq 0$ and $\tau = -1$, $z_0$ satisfies the following equation and elements characteristics:

$$((x^2 - 1)(x^2 - \upsilon^2)z_0) + \left( -5x^3 - x^2 + (3 + 2\upsilon^2)x + \upsilon^2 \right)z_0 = 0,$$

where

$$(z_0)_1 = -1, \quad (z_0)_2 = \frac{1}{4}(\upsilon^2 + 3),$$

and

$$\beta_n = (-1)^{n+1}, \quad \gamma_{n+1} = \frac{\upsilon^2 - 1}{4}, \quad \upsilon^2 \neq 1, \upsilon \neq 0, n \geq 0.$$ (3.44)

This form is of class $s = 2$. Indeed, through a suitable shifting, we apply the operator $h_{-1}$ in (3.42), (3.43), and (3.44). We obtain the previous case.

Likewise, if $\upsilon \neq 0$ and $\tau = \upsilon$, then $z_0$ is given by

$$((x^2 - 1)(x^2 - \upsilon^2)z_0) + \left( -5x^3 + \upsilon x^2 + (2 + 3\upsilon^2)x - \upsilon \right)z_0 = 0,$$

where

$$(z_0)_1 = \upsilon, \quad (z_0)_2 = \frac{1}{4}(3\upsilon^2 + 1),$$

and

$$\beta_n = \frac{(-1)^n}{\upsilon}, \quad \gamma_{n+1} = \frac{1 - \upsilon^2}{4}, \quad \upsilon^2 \neq 1, \upsilon \neq 0, n \geq 0.$$ (3.47)

Applying the operator $h_{\upsilon}$ in (3.45) and (3.47), then while replacing $\upsilon$ by $\upsilon^{-1}$, we obtain again case (b).

By a similar calculation, if $\upsilon \neq 0$ and $\tau = -\upsilon$, then $z_0$ is given by

$$((x^2 - 1)(x^2 - \upsilon^2)z_0) + \left( -5x^3 - \upsilon x^2 + (2 + 3\upsilon^2)x + \upsilon \right)z_0 = 0,$$

where

$$(z_0)_1 = -\upsilon, \quad (z_0)_2 = \frac{1}{4}(3\upsilon^2 + 1),$$

and

$$\beta_n = (-1)^{n+1}\upsilon, \quad \gamma_{n+1} = \frac{1 - \upsilon^2}{4}, \quad \upsilon^2 \neq 1, \upsilon \neq 0, n \geq 0.$$ (3.50)

Applying the operator $h_{-\upsilon}$ in (3.48) and (3.50), then while replacing $\upsilon$ by $\upsilon^{-1}$, we obtain again case (b).
(c) In this case, we have

\[(x^2(x - \tau)(x^2 - 1)z_0)' - 3x(x - \tau)(2x^2 - 1)z_0 = 0.\]  \(3.51\)

From Table 3.1, \(\Psi(0) + \Phi'(0) = 0, \langle z_0, \theta \Psi + \theta_0^2 \Phi \rangle = 0, \) and \(|\Psi(c) + \Phi'(c)| + |\langle z_0, \theta \Psi + \theta_0^2 \Phi \rangle| \neq 0 \) for all \(c \in \{-1, 1, \tau\} \).

Then this equation is simplified by \(x\), and \(z_0\) satisfies \((\Phi_1 z_0)' + \Psi_1 z_0 = 0, \) where

\[\Phi_1(x) = x(x - \tau)(x^2 - 1), \quad \Psi_1(x) = (x - \tau)(-5x^2 + 2).\]  \(3.52\)

From Lemma 2.3, \(\Psi_1(c) + \Phi'_1(c)| + |\langle z_0, \theta \Psi_1 + \theta_0^2 \Phi_1 \rangle| \neq 0 \) for all \(c \in \{-1, 1, \tau\} \); and taking into account \(\Psi_1(0) + \Phi'_1(0) = -\tau \neq 0, \) we deduce the result.

(d) From Table 3.1, the equation \((x^2(x - 1)(x^2 - 1)z_0)' - 3x(x - 1)(2x^2 - 1)z_0 = 0\) is simplified twice by \(x\) and \(x - 1\). In the first place, we have

\[(x(x - 1)(x^2 - 1)z_0)' + (x - 1)(-5x^2 + 2)z_0 = 0.\]  \(3.53\)

Next, we simplify once more by \(x - 1\), and we have \((\Phi_2 z_0)' + \Psi_2 z_0 = 0, \) where

\[\Phi_2(x) = x(x^2 - 1), \quad \Psi_2(x) = -4x^2 + x + 2.\]  \(3.54\)

Then we get \(\Psi_2(0) + \Phi'_2(0) = 1 \neq 0, \) and according to Lemma 2.3, \(z_0\) is a semiclassical form of class \(s = 1, \) which satisfies \(3.35.\)

If \(\nu = 0\) and \(\tau = -1, \) \(z_0\) is given by

\[\beta_n = (-1)^{n+1}, \quad \gamma_{n+1} = -\frac{1}{4}, \quad n \geq 0.\]  \(3.55\)

This form is of class \(s = 1.\) In fact, applying the operator \(h_{-1}\) in \(3.55, \) we have again case (d).

(e) Similarly, from Table 3.1, it is easy to prove that the equation is simplified by \(x^3. \) Therefore, \(z_0\) is a classical form given by \(3.36. \) \(\square\)

4. Quadratic decomposition of the second-order self-associated orthogonal sequences

In order to build a structure relation and a differential equation related to second-order self-associated sequences, we want their quadratic decomposition given by \(2.28. \) In [9],
The second-order self-associated orthogonal sequences

the first author gave necessary and sufficient conditions for the sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) to be orthogonal.

**Proposition 4.1.** Let \( \{W_n\}_{n \geq 0} \) satisfy the recurrence relation (2.20), where

\[
\beta_n = (-1)^n \beta_0, \quad n \geq 0. \quad (4.1)
\]

Then there exist two MOPSs \( \{P_n\}_{n \geq 0} \) with respect to \( u_0 \), and \( \{R_n\}_{n \geq 0} \), with respect to \( v_0 \), fulfilling the following relations:

\[
P_n(x) = 1, \quad P_1(x) = x - y_1 - \beta_0^2, \quad \quad (4.2)
\]

\[
P_{n+2}(x) = (x - y_{2n+2} - y_{2n+3} - \beta_0^2)P_{n+1}(x) - y_{2n+1}y_{2n+2}P_n(x), \quad n \geq 0,
\]

\[
R_n(x) = 1, \quad R_1(x) = x - y_2 - \beta_0^2, \quad \quad (4.3)
\]

\[
R_{n+2}(x) = (x - y_{2n+3} - y_{2n+4} - \beta_0^2)R_{n+1}(x) - y_{2n+2}y_{2n+3}R_n(x), \quad n \geq 0,
\]

\[
P_{n+1}(x) = R_{n+1}(x) + y_{2n+2}R_n(x), \quad n \geq 0, \quad (4.4)
\]

\[
(x - \beta_0^2)R_n(x) = P_{n+1}(x) + y_{2n+1}P_n(x), \quad n \geq 0, \quad (4.5)
\]

since, in (2.28), \( a_n(x) = 0 \) and \( b_n(x) = -\beta_0 R_n(x), n \geq 0. \)

Moreover, the forms \( u_0, v_0, \) and \( w_0 \) satisfy

\[
u_0 = \sigma w_0, \quad (4.6)
\]

\[
\sigma(xw_0) = \beta_0 (\sigma w_0), \quad (4.7)
\]

\[
v_0 = \frac{1}{\gamma_1} (x - \beta_0^2) (\sigma w_0). \quad (4.8)
\]

Now, this result will be applied to \( \{Z_n\}_{n \geq 0} \) which, by virtue of (3.24), fulfils (4.1) and

\[
Z_{2n}(x) = P_n(x^2), \quad (4.9)
\]

\[
Z_{2n+1}(x) = (x - \tau)R_n(x^2). \quad (4.10)
\]

From (3.24) and (4.2), the sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) become

\[
P_0(x) = 1, \quad P_1(x) = x - \frac{1}{4} (1 + v^2 + 2\tau^2) - \frac{1}{2} \epsilon \gamma_{r,v}, \quad (4.11)
\]

\[
P_{n+2}(x) = \left( x - \frac{1}{2} (1 + v^2) \right) P_{n+1}(x) - \left( \frac{v^2 - 1}{4} \right) P_n(x), \quad n \geq 0,
\]

\[
R_0(x) = 1, \quad R_1(x) = x - \frac{1}{2} (1 + v^2), \quad (4.12)
\]

\[
R_{n+2}(x) = \left( x - \frac{1}{2} (1 + v^2) \right) R_{n+1}(x) - \left( \frac{v^2 - 1}{4} \right) R_n(x), \quad n \geq 0.
\]
We remark that the sequence $\{P_n\}_{n \geq 0}$ is the corecursive sequence of $\{R_n\}_{n \geq 0}$ with the value $-\gamma_2 = -\left(\frac{1}{4}\right)(1 + \nu^2 - 2\tau^2) + (1/2)\varepsilon \tau,\nu$. For the parameter $P_n(x) = R_n(-\gamma_2; x)$, $n \geq 0$, we have

$$P_{n+1} = R_{n+1} + \gamma_2 R_n^{(1)} = R_{n+1} + \gamma_2 R_n, \quad n \geq 0,$$

(4.13)
in accordance with (4.4). Moreover, (4.5) becomes

$$(x - \tau^2) R_n(x) = P_{n+1}(x) + \gamma_1 P_n(x), \quad n \geq 0.$$  

(4.14)

From (4.12), we easily see that

$$R_n(x) = a^n P_n^{(1/2,1/2)}(a^{-1}(x - b)), \quad n \geq 0, \quad a = \frac{1}{2}(\nu^2 - 1), \quad b = \frac{1}{2}(1 + \nu^2),$$

(4.15)

where $\{\hat{P}_n^{(a,b)}\}_{n \geq 0}$ is the monic Jacobi polynomials sequence, orthogonal with respect to the Jacobi form $\mathcal{J}(\alpha, \beta)$, with parameters $\alpha, \beta$, see [11, 12], fulfilling the following equation:

$$((x^2 - 1)\mathcal{J}(\alpha, \beta))' + (-(\alpha + \beta + 2)x + \alpha - \beta)\mathcal{J}(\alpha, \beta) = 0, \quad (\mathcal{J}(\alpha, \beta))_0 = 1.$$

(4.16)

Usually, $\mathcal{J}(1/2,1/2)$ is denoted by $\mathcal{U}$ which fulfils (3.36), and $\{\hat{P}_n^{(1/2,1/2)}(x)\}_{n \geq 0}$ is defined by (3.37).

Since $v_0 = (\tau_b \circ h_{a})\mathcal{U}$, we have

$$(\Phi_0 v_0)' + \Psi_0 v_0 = 0,$$

(4.17)

where

$$\Phi_0(x) = (x - 1)(x - \nu^2), \quad \Psi_0(x) = -\frac{3}{2}(2x - 1 - \nu^2).$$

(4.18)

Likewise, from (4.6) and (4.8), taking (4.17) into account, we obtain

$$(\Phi_1 u_0)' + \Psi_1 u_0 = 0,$$

(4.19)

$$\left(u_0\right)_1 = (\sigma z_0)_1 = \tau^2 + \gamma_1 = \frac{1}{4}(1 + \nu^2 + 2\tau^2) + \frac{1}{2}\varepsilon \tau,\nu,$$

where

$$\Phi_1(x) = (x - 1)(x - \nu^2)(x - \tau^2), \quad \Psi_1(x) = -\frac{3}{2}(2x - 1 - \nu^2)(x - \tau^2).$$

(4.20)
Lemma 4.2. The following cases hold:

(a) if $\tau^2 \neq 1$ and $\tau^2 \neq \nu^2$, the class of $u_0$ is $s = 1$;
(b) if $\tau^2 = 1$ and $\tau^2 \neq \nu^2$, the form $u_0$ is classical ($s = 0$) and fulfils the equation

$$((x-1)(x-\nu^2)u_0)' - \frac{1}{2}(4x-3-\nu^2)u_0 = 0, \quad (u_0)_1 = \frac{1}{4}(3+\nu^2); \quad (4.21)$$

this implies

$$u_0 = (\tau_b \circ h_a)\mathcal{J}\left(-\frac{1}{2}, \frac{1}{2}\right) \quad (4.22)$$

with

$$a = \frac{1}{2}(\nu^2 - 1), \quad b = \frac{1}{2}(1+\nu^2); \quad (4.23)$$

(c) if $\tau^2 = \nu^2$, the form $u_0$ is classical and fulfils the equation

$$((x-1)(x-\tau^2)u_0)' - \frac{1}{2}(4x-1-3\tau^2)u_0 = 0, \quad (u_0)_1 = \frac{1}{4}(1+3\tau^2); \quad (4.24)$$

this implies

$$u_0 = (\tau_b \circ h_a)\mathcal{J}\left(\frac{1}{2}, -\frac{1}{2}\right) \quad (4.25)$$

with

$$a = \frac{1}{2}(\tau^2 - 1), \quad b = \frac{1}{2}(1+\tau^2). \quad (4.26)$$

Proof. From (4.20), we have

$$\Phi'_1(1) + \Psi_1(1) = -\frac{1}{2}(1-\nu^2)(1-\tau^2),$$

$$\Phi'_1(\nu^2) + \Psi_1(\nu^2) = -\frac{1}{2}(\nu^2-1)(\tau^2-\nu^2), \quad (4.27)$$

$$\Phi'_1(\tau^2) + \Psi_1(\tau^2) = (\tau^2-1)(\tau^2-\nu^2).$$

Assertion (a) is evident. When $\tau^2 = 1$ and $\tau^2 \neq \nu^2$, we have

$$\langle u_0, \theta_1^2\Phi_1 + \theta_1\Psi_1 \rangle = \left\langle u_0, -2x + \frac{1}{2}(3+\nu^2) \right\rangle = -2(u_0)_1 + \frac{1}{2}(3+\nu^2) = 0, \quad (4.28)$$

whence (4.21) and (4.22). The same applies to (4.24) and (4.25). \qed
5. Structure relation and differential equation

It is well known that a semiclassical orthogonal polynomials sequence fulfills a second-order differential equation [3, 5, 10]. In this section, we give the following second-order differential equation fulfilled by \( \{Z_n\}_{n \geq 0} \). We have

\[
J(x; n)Z''_{n+1}(x) + K(x; n)Z'_{n+1}(x) + L(x; n)Z_{n+1}(x) = 0, \quad n \geq 0,
\]

with

\[
J(x; n) = \Phi(x)D_{n+1}(x), \quad n \geq 0,
\]

\[
K(x; n) = C_0(x)D_{n+1}(x) - W(\Phi, D_{n+1})(x), \quad n \geq 0,
\]

\[
L(x; n) = W\left( \frac{1}{2} (C_{n+1} - C_0), D_{n+1} \right)(x) - D_{n+1}(x) \sum_{\nu=0}^{n} D_{\nu}(x), \quad n \geq 0,
\]

where \( W(f, g) = fg' - gf' \) is the Wronskian of \( f \) and \( g \).

The sequences \( \{C_n\}_{n \geq 0} \) and \( \{D_n\}_{n \geq 0} \) are defined by

\[
\Phi(z)S'(z^{(n)}_0)(z) = B_n(z)S^2(z^{(n)}_0)(z) + C_n(z)S(z^{(n)}_0)(z) + D_n(z), \quad n \geq 0,
\]

and fulfill

\[
B_0(z) = 0,
\]

\[
C_0(z) = -\Phi'(z) - \Psi(z),
\]

\[
D_0(z) = -(z_0 \theta_0 \Phi)'(z) - (z_0 \theta_0 \Psi)(z),
\]

\[
B_{n+1}(z) = \gamma_{n+1}D_n(z), \quad n \geq 0,
\]

\[
C_{n+1}(z) = -C_n(z) + 2(z - \beta_n)D_n(z), \quad \text{deg} C_n \leq 4, \quad n \geq 0,
\]

\[
y_{n+1}D_{n+1}(z) = -\Phi(z) + B_n(z) - (z - \beta_n)C_n(z) + (z - \beta_n)^2D_n(z), \quad \text{deg} D_n \leq 3, \quad n \geq 0.
\]

They are involved in the so-called structure relation [3, 10]

\[
\Phi(x)Z'_{n+1}(x) = \frac{1}{2} (C_{n+1}(x) - C_0(x))Z_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)Z_n(x), \quad n \geq 0.
\]

Here, from (3.22), (3.23), and (5.4), we have

\[
\Phi(z) = (z - \tau)(z^2 - 1)(z^2 - \nu^2),
\]

\[
C_0(z) = z^4 - 2\tau z^3 + \tau (1 + \nu^2)z - \nu^2,
\]

\[
D_0(z) = 2z \left( z^2 + 2\gamma_1 - \frac{1}{2} (1 + \nu^2) \right) = 2z (z^2 - \tau^2 + \epsilon \sigma_{\tau, \nu}).
\]
Indeed, from (2.2), we have

\[
(z_0 \theta_0 \Phi)(x) = \left\langle z_0, \frac{\Phi(x) - \Phi(\xi)}{x - \xi} \right\rangle
= \left\langle z_0, (x - \tau)(x^4 - (1 + \nu^2)x^2 + \nu^2) - (\xi - \tau)(\xi^4 - (1 + \nu^2)(\xi^2 + \nu^2)) \right\rangle
= \left\langle z_0, x^4 + (\xi - \tau)x^3 + (\xi^2 - (1 + \nu^2)\xi - \tau\xi)x^2
+ (\xi^3 - (1 + \nu^2)\xi - \tau\xi^2 + (1 + \nu^2)\tau)x
+ \xi^4 - \tau\xi^3 - (1 + \nu^2)\xi + \tau(1 + \nu^2)\xi + \nu^2 \right\rangle
= x^4 + ((z_0)_1 - \tau)x^3 + ((z_0)_2 - (1 + \nu^2) - \tau(z_0)_1)x^2
+ ((z_0)_3 - \tau(z_0)_2 - (1 + \nu^2)((z_0)_1 - \tau))x
+ (z_0)_4 - \tau(z_0)_3 - (1 + \nu^2)(z_0)_1 + \tau(1 + \nu^2)(z_0)_1 + \nu^2.
\]

Through (3.25), \((z_0)_1 = \tau, (z_0)_2 = \gamma_1 + \tau^2, \) and \((z_0)_3 = \tau(z_0)_2; \) so

\[
(z_0 \theta_0 \Phi)'(x) = 4x^3 + 2(\gamma_1 - (1 + \nu^2))x.
\]  

(5.9)

In the same way, from (2.2) and (3.23), we get

\[
(z_0 \theta_0 \Psi)(x) = \left\langle z_0, -6x^3 + (6\tau - 6\xi)x^2 + (6\tau\xi - 6\xi^2 + 3(1 + \nu^2))x
- 6\xi^3 + 6\tau\xi^2 + 3(1 + \nu^2)(\xi - \tau) \right\rangle
= -6x^3 + (3(1 + \nu^2) - 6\gamma_1)x.
\]

(5.10)

Thus, we deduce the expression of \(D_0(x).\)

Generally, it is difficult to give the sequences \(\{C_n\}_{n \geq 0}\) and \(\{D_n\}_{n \geq 0}\) explicitly using the recurrence relations (5.5). The quadratic decomposition allows us to do it.

**Lemma 5.1.** The following structure relations hold:

\[
(x - 1)(x - \nu^2)R'_{n+1}(x) = (n + 1) \left( x - \frac{1}{2}(1 + \nu^2) \right) R_{n+1}(x)
- 2(n + 2) \left( \frac{1 - \nu^2}{4} \right) R_n(x), \quad n \geq 0,
\]  

(5.11)

\[
\Phi_1(x)P'_{n+1}(x) = A(n;x)P_{n+1}(x) - B(n;x)P_n(x), \quad n \geq 0,
\]

(5.12)
where

\[ \Phi_1(x) = (x - 1)(x - v^2)(x - r^2), \quad (5.13) \]

\[ A(n; x) = (n + 1)\left( x + 2y_2 - \frac{1}{2}(v^2 + 1) \right) \left( x + y_1 - \frac{1}{2}(v^2 + 1) \right) \]
\[ - (n + 2)y_2 \left( x + 2y_1 - \frac{1}{2}(v^2 + 1) \right), \quad n \geq 0, \quad (5.14) \]

\[ B(n; x) = y_1y_2 \left\{ (n + 1) \left( x + 2y_2 - \frac{1}{2}(v^2 + 1) \right) \right\} \]
\[ + (n + 2) \left( x + 2y_1 - \frac{1}{2}(v^2 + 1) \right), \quad n \geq 0. \quad (5.15) \]

**Proof.** Since, for the Jacobi sequence, we have [10, 11]

\[ C_n^{(\alpha, \beta)}(x) = (2n + \alpha + \beta)x - \frac{\alpha^2 - \beta^2}{2n + \alpha + \beta}, \quad n \geq 0, \quad (5.16) \]

\[ D_n^{(\alpha, \beta)}(x) = 2n + \alpha + \beta + 1, \quad n \geq 0, \]

then, in the case \( \alpha = \beta = 1/2 \), we obtain

\[ C_n^R(x) = aC_n^{(1/2, 1/2)} \left( \frac{x - b}{a} \right) = (2n + 1) \left( x - \frac{1}{2}(1 + v^2) \right), \quad n \geq 0, \quad (5.17) \]

\[ D_n^R(x) = D_n^{(1/2, 1/2)} \left( \frac{x - b}{a} \right) = 2n + 2, \quad n \geq 0, \]

where \( a = (1/2)(v^2 - 1) \) and \( b = (1/2)(1 + v^2) \).

Hence, (5.11) holds.

Next, from (4.4), we have

\[ \Phi_1(x)P_{n+1}'(x) = (x - 1)(x - v^2)(x - r^2)R_{n+1}'(x) \]
\[ + y_2(x - 1)(x - v^2)(x - r^2)R_n'(x), \quad n \geq 0. \quad (5.18) \]

According to (5.11) and taking (4.12) into account, we obtain

\[ \Phi_1(x)P_{n+1}'(x) = (n + 1) \left( x + 2y_1 - \frac{1}{2}(v^2 + 1) \right) \left( x - r^2 \right) R_{n+1}(x) \]
\[ - (n + 2) \left( y_2 \left( x - \frac{1}{2}(v^2 + 1) \right) + 2y_1y_2 \right) \left( x - r^2 \right) R_n(x), \quad n \geq 0. \quad (5.19) \]

With (4.5), this yields (5.12), (5.13), (5.14), and (5.15). \( \square \)
Proposition 5.2. The sequence \( \{Z_n\}_{n\geq 0} \) fulfils (5.6), where the sequences \( \{C_n\}_{n\geq 0} \) and \( \{D_n\}_{n\geq 0} \) are given by

\[
C_{2n}(x) = (4n + 1)x^4 - 2\tau(2n + 1)x^3 + 4n\left(\frac{1}{2}(v^2 + 1) - 2(y_1 + \tau^2)\right)x^2
+ \tau(8(\tau^2 + y_1)n - (2n - 1)(1 + v^2))x - v^2, \quad n \geq 0,
\]

\[
D_{2n}(x) = 2x\left(2n + 1\right)x^2 - 2n\tau^2 + 2y_1 - \frac{1}{2}(v^2 + 1), \quad n \geq 0,
\]

\[
C_{2n+1}(x) = (4n + 3)x^4 - 2\tau(2n + 1)x^3 + 2(n + 1)(4y_1 - (v^2 + 1))x^2
- 2\tau\left(4y_1(n + 1) - \frac{1}{2}(2n + 1)(v^2 + 1)\right)x + v^2, \quad n \geq 0,
\]

\[
D_{2n+1}(x) = 4(n + 1)x(x - \tau)^2, \quad n \geq 0.
\]

Proof. We start with (5.11), where \( x \to x^2 \). According to

\[
Z'_{2n+3}(x) = R_{n+1}(x^2) + 2x(x - \tau)R'_{n+1}(x^2), \quad n \geq 0,
\]

obtained by differentiating (4.10), relation (5.11) becomes

\[
\Phi(x)Z'_{2n+3}(x) = \left((x^2 - 1)(x^2 - v^2) + 2(n + 1)x(x - \tau)\left(x^2 - \frac{1}{2}(v^2 + 1)\right)\right)Z_{2n+3}(x)
- 4\left(\frac{1 - v^2}{4}\right)^2(n + 2)x(x - \tau)^2 R_n(x^2), \quad n \geq 0.
\]

But (4.9) and (4.13) provide

\[
\Phi(x)Z'_{2n+3}(x) = E(n; x)Z_{2n+3}(x) - 4y_1(n + 2)x(x - \tau)^2 Z_{2n+2}(x), \quad n \geq 0,
\]

where

\[
E(n; x) = (x^2 - 1)(x^2 - v^2) + 2x(x - \tau)\left((n + 1)\left(x^2 - \frac{1}{2}(v^2 + 1)\right) + 2(n + 2)y_1\right).
\]

Comparing (5.26) with (5.6), where \( n \to 2n + 2 \), leads to

\[
\left(E(n; x) - \frac{1}{2}(C_{2n+3}(x) - C_0(x))\right)Z_{2n+3}(x)
= y_1(4(n + 2)x(x - \tau)^2 - D_{2n+3}(x))Z_{2n+2}(x), \quad n \geq 0.
\]
This yields
\[ \frac{1}{2} (C_{2n+1}(x) - C_0(x)) = E(n-1;x), \quad n \geq 1, \tag{5.29} \]
\[ D_{2n+1}(x) = 4(n+1)x(x-\tau)^2, \quad n \geq 1, \]
by virtue of a well-known result on orthogonal sequences. Routine calculation from (5.5) shows that (5.29) is valid for \( n \geq 0 \), whence (5.22) and (5.23).

Next, from (5.12), where \( x \to x^2 \), and with (4.9), we obtain
\[ (x+\tau)\Phi(x)Z'_{2n+2}(x) = 2xA(n;x^2)Z_{2n+2}(x) - 2xB(n;x^2)Z_{2n}(x). \tag{5.30} \]

But
\[ Z_{2n}(x) = \frac{1}{\gamma_1} (x+\tau)Z_{2n+1}(x) - \frac{1}{\gamma_1} Z_{2n+2}(x) \tag{5.31} \]
implies
\[ (x+\tau)\Phi(x)Z'_{2n+2}(x) = 2x(A(n;x^2) + \gamma_1^{-1}B(n;x^2))Z_{2n+2}(x) \]
\[ - 2\gamma_1^{-1}x(x+\tau)B(n;x^2)Z_{2n+1}(x). \tag{5.32} \]

Taking (5.14) and (5.15) into account, we have
\[ A(n;x^2) + \gamma_1^{-1}B(n;x^2) = (n+1)(x^2 - \tau^2) \left( x^2 + 2\gamma_2 - \frac{1}{2}(\upsilon^2 + 1) \right). \tag{5.33} \]

This leads to
\[ \Phi(x)Z'_{2n+2}(x) = 2(n+1)x(x-\tau) \left( x^2 + 2\gamma_2 - \frac{1}{2}(\upsilon^2 + 1) \right) Z_{2n+2}(x) \]
\[ - 2\gamma_2 x \left( (n+1)(x^2 + 2\gamma_2 - \frac{1}{2}(\upsilon^2 + 1)) + (n+2)(x^2 + 2\gamma_1 - \frac{1}{2}(\upsilon^2 + 1)) \right) Z_{2n+1}(x), \quad n \geq 0. \tag{5.34} \]

As above, we obtain
\[ C_{2n}(x) = C_0(x) + 4nx(x-\tau) \left( x^2 + 2\gamma_2 - \frac{1}{2}(\upsilon^2 + 1) \right), \]
\[ D_{2n}(x) = 2x \left( n \left( x^2 + 2\gamma_2 - \frac{1}{2}(\upsilon^2 + 1) \right) + (n+1)(x^2 + 2\gamma_1 - \frac{1}{2}(\upsilon^2 + 1)) \right), \quad n \geq 2. \tag{5.35} \]

In fact, these relations are valid for \( n \geq 0 \), whence (5.20) and (5.21).

Now, we are able to calculate the coefficients of (5.1) defined by (5.2). □
Proof. From (5.2), (5.7), (5.21), and (5.23), it is easy to obtain (5.36) and (5.37). Next, we have

\[ J(x; 2n) = 4(n + 1)x(x - \tau)^3(x^2 - 1)(x^2 - v^2), \]
\[ (5.36) \]
\[ J(x; 2n + 1) = 2x(x - \tau)(x^2 - 1)(x^2 - v^2) \left\{ (2n + 3)x^2 - 2(n + 1)\tau^2 + 2\gamma_1 - \frac{1}{2}(v^2 + 1) \right\}, \]
\[ (5.37) \]
\[ K(x; 2n) = 4(n + 1)(x - \tau)^2 \left\{ 3x^5 - 5\tau x^4 + 2\tau(1 + v^2)x^2 - 3v^2x + \tau v^2 \right\}, \quad n \geq 0, \]
\[ (5.38) \]
\[ K(x; 2n + 1) = (x - \tau) \left\{ 3(4n + 6)x^6 - (20(n + 1)\tau^2 - 5(4\gamma_1 - (v^2 + 1)))x^4 
\right.
\[ + \left\{ (1 + v^2)(8(n + 1)\tau^2 - 2(4\gamma_1 - (v^2 + 1))) - 3(4n + 6)v^2 \right\}x^2 
\right.
\[ + \left( 4n + 1 \right)\tau^2 v^2 - v^2(4\gamma_1 - (v^2 + 1)) \}, \quad n \geq 0, \]
\[ (5.39) \]
\[ L(x; 2n) = -4(n + 1)(x - \tau) \left\{ (2n + 1)(2n + 3)x^5 - (8n^2 + 16n + 5)\tau x^4 
\right.
\[ + 4n(n + 2)\tau^2 x^3 + 2(1 + v^2)\tau x^2 
\right.
\[ - 3v^2x + \tau v^2 \}, \quad n \geq 0, \]
\[ (5.40) \]
\[ L(x; 2n + 1) = -4(n + 1)(n + 2)x^2 \left\{ 2(2n + 3)x^4 - 2(2n + 3)\tau x^3 
\right.
\[ + (3(4\gamma_1 - (v^2 + 1)) - 4n\tau^2)x^2 - (4\gamma_1 - (v^2 + 1)) 
\right.
\[ + 4(n + 2)\tau^2 \tau x \}, \quad n \geq 0. \]
\[ (5.41) \]
Successively, we get

\[
\frac{1}{2} (C_{2n+1} - C_0) (x) = E(n-1; x) = (x^2 - 1)(x^2 - v^2) + 2x(x - \tau) \left\{ n\left(x^2 - \frac{1}{2}(v^2 + 1)\right) + 2(n+1)\gamma_1 \right\},
\]

\[
\frac{1}{2} (C_{2n+1} - C_0) (x) D'_{2n+1}(x)
\]

\[
= 4(n+1)(x-\tau)(3x-\tau)\left\{ (2n+1)x^4 - 2n\tau x^3 + (n+1)(4\gamma_1 - (v^2 + 1))x^2 - \tau(4(n+1)\gamma_1 - n(1 + v^2))x + v^2 \right\}
\]

\[
= 4(n+1)(x-\tau)\left\{ 3(2n+1)x^5 - (8n+1)\tau x^4 + (3(n+1)(4\gamma_1 - (v^2 + 1)) + 2n\tau^2)x^3 - \tau(16(n+1)\gamma_1 - (4n+1)(1 + v^2))x^2 + (\tau^2(4(n+1)\gamma_1 - n(1 + v^2)) + 3v^2)x - \tau v^2 \right\}.
\]

Next

\[
\frac{1}{2} (C_{2n+1} - C_0)' (x) D_{2n+1}(x)
\]

\[
= 8(n+1)x(x-\tau)^2 \left\{ 2(2n+1)x^3 - 3n\tau x^2 + (n+1)(4\gamma_1 - (v^2 + 1))x - \tau\left( 2(n+1)\gamma_1 - \frac{1}{2}(1 + v^2)\right) \right\}
\]

\[
= 4(n+1)(x-\tau) \left\{ 4(2n+1)x^5 - 2(7n+2)\tau x^4 + 2((n+1)(4\gamma_1 - (v^2 + 1)) + 3n\tau^2)x^3 - 2\tau\left( 6(n+1)\gamma_1 - \frac{1}{2}(2n+1)(1 + v^2)\right)x^2 + 2\tau^2\left( 2(n+1)\gamma_1 - \frac{1}{2}n(1 + v^2)\right)x \right\}.
\]

Further, since

\[
\sum_{\gamma=0}^{2n} D_\gamma(x) = \sum_{\gamma=0}^{n} D_{2\gamma}(x) + \sum_{\gamma=0}^{n-1} D_{2\gamma+1}(x),
\]

\[
\sum_{\gamma=0}^{n} D_{2\gamma}(x) = 2(n+1)x\left( (n+1)x^2 + \left( 2\gamma_1 - \frac{1}{2}(v^2 + 1) - n\tau^2 \right) \right),
\]

\[
\sum_{\gamma=0}^{n-1} D_{2\gamma+1}(x) = 2n(n+1)x(x-\tau)^2,
\]

(5.44)
The second-order self-associated orthogonal sequences

we obtain

\[
D_{2n+1}(x) \sum_{y=0}^{2n} D_y(x)
= 4(n+1)^2(x-\tau)\{2(2n+1)x^5 - 2(4n+1)\tau x^4
+ (4y_1 - (v^2 + 1) + 4n\tau^2)x^3 - (4y_1 - (v^2 + 1))\tau x^2\}.
\]

(5.47)

This leads to (5.40). Similar calculations can be used to prove (5.41).

6. The integral representations of the second-order self-associated forms

Throughout this section, we will suppose \( \nu \in \mathbb{R} - \{-1,1\} \). It will be sufficient to consider \( 0 \leq \nu < 1 \) or \( \nu > 1 \).

From (3.19), the formal Stieltjes function \( S(z_0) \) is given by

\[
S(z_0)(z) = \frac{1}{2} y_2^{-1}(z-\tau)^{-1}\{(z^2 - 1)^{1/2}(z^2 - \nu^2)^{1/2} - 2y_2 - W(z)\}
\]

(6.1)

with \( W(z) = z^2 - (1/2)(\nu^2 + 1), z_0 = z_0(\tau, v, \varepsilon), \) and \( y_2 = y_2(\tau, v, \varepsilon) \). Putting

\[
w(\tau) = w(\tau, v, \varepsilon) = (x-\tau)z_0(\tau, v, \varepsilon),
\]

(6.2)

we have \( S(w(\tau))(z) = (z-\tau)S(z_0)(z) + 1 \). Therefore, taking (6.1) into account, we get

\[
S(w(\tau, v, \varepsilon))(z) = \frac{1}{2} y_2^{-1}Q(z),
\]

(6.3)

where

\[
Q(z) = (z^2 - 1)^{1/2}(z^2 - \nu^2)^{1/2} - W(z).
\]

(6.4)

Since \( y_2(\tau, v, -\varepsilon) = y_1(\tau, v, \varepsilon) \), we have

\[
S(w(\tau, v, -\varepsilon))(z) = \frac{1}{2} y_1^{-1}Q(z).
\]

(6.5)

Consequently, it is sufficient to study the case \( \varepsilon = 1 \).

Choosing the branch which is positive when \( z^2 - 1 > 0 \) and \( z^2 - \nu^2 > 0 \), we see that \( Q \) is regular in the upper half-plane. Moreover, it is easy to prove

\[
\sup_{y > 0} \int_{-\infty}^{+\infty} |Q(x + iy)|^2 dx < +\infty.
\]

(6.6)

Consequently, the function \( Q \) possesses the following representation [2]:

\[
Q(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Q(t + i0)}{t - z} dt, \quad \Im z > 0.
\]

(6.7)
We obtain from (6.4) that

(i) for $0 \leq \nu < 1$,

$$
\Im Q(x + i0) = \begin{cases} 
0, & |x| > 1, \\
\text{sgn} x \sqrt{(1 - x^2)(x^2 - \nu^2)}, & |x| < 1, \\
\nu < |x| < 1,
\end{cases}
$$

(ii) for $\nu > 1$,

$$
\Im Q(x + i0) = \begin{cases} 
0, & |x| > \nu, \\
\text{sgn} x \sqrt{(x^2 - 1)(\nu^2 - x^2)}, & 1 < |x| < \nu,
\end{cases}
$$

In accordance with (6.3), this leads to

$$
\langle w(\tau), f \rangle = \frac{1}{2\pi \gamma^2} \int_{-\nu}^{+\nu} \Im Q(x + i0) f(x) dx, \quad f \in \mathcal{F},
$$

where

$$
\nu := \max(1, \nu).
$$

But from (6.2), we have

$$
z_0 = \delta_\tau + (x - \tau)^{-1} z(\tau).
$$

This yields

$$
\langle z_0, f \rangle = f(\tau) + \frac{1}{2\pi \gamma^2} \int_{-\nu}^{+\nu} \frac{\Im Q(x + i0) f(x) - f(\tau)}{x - \tau} dx.
$$

When $\tau \in \mathbb{C} - [ -\nu, +\nu]$, we get

$$
\langle z_0, f \rangle = \left\{ 1 - \frac{1}{2\pi \gamma^2} \int_{-\nu}^{+\nu} \frac{\Im Q(x + i0)}{x - \tau} dx \right\} f(\tau) + \frac{1}{2\pi \gamma^2} \int_{-\nu}^{+\nu} \frac{\Im Q(x + i0)}{x - \tau} f(x) dx.
$$

On account of (6.4) and (6.7), we obtain

$$
(\tau^2 - 1)^{1/2} (\tau^2 - \nu^2)^{1/2} - \tau^2 + \frac{1}{2} (\nu^2 + 1) = \frac{1}{\pi} \int_{-\nu}^{+\nu} \frac{\Im Q(t + i0)}{t - \tau} dt.
$$

But $2\gamma_1 = (\tau^2 - 1)^{1/2} (\tau^2 - \nu^2)^{1/2} - \tau^2 + 1/2(\nu^2 + 1)$; accordingly, (6.14) becomes

$$
\langle z_0, f \rangle = (1 - \gamma_1 \gamma_2^{-1}) f(\tau) + \frac{1}{2\pi \gamma^2} \int_{\nu < |x| < \nu} \frac{\text{sgn} x \sqrt{(\nu^2 - x^2)(x^2 - \nu^2)}}{x - \tau} f(x) dx,
$$

where $\nu := \min(1, \nu)$. 

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When $\tau \in ]-\upsilon, \upsilon[$, we distinguish two cases.

(a) $\upsilon \leq |\tau| < \upsilon$. From (6.13), we have

$$\langle z_0, f \rangle = f(\tau) + \frac{1}{2\pi y_2} \int_{|y|<|\tau|} \mathfrak{Q}(x + i0) \frac{f(x) - f(\tau)}{x - \tau} \, dx$$

(6.17)

with

$$y_2(\tau) = \frac{1}{2}(1 + \upsilon^2) - \tau^2 - \frac{1}{2} Q(\tau + i0).$$

(6.18)

It is easy to see that

$$\Re Q(x + i0) = \begin{cases} \sqrt{(x^2 - \upsilon^2)(x^2 - \bar{\upsilon}^2)} - W(x), & |x| > \upsilon, \\ -W(x), & \upsilon \leq |x| < \upsilon, \\ -\sqrt{(\upsilon^2 - x^2)(\bar{\upsilon}^2 - x^2)} - W(x), & |x| < \upsilon. \end{cases}$$

(6.19)

Consequently,

$$y_2(\tau) = -\frac{1}{2}(W(\tau) + i \text{sgn} \tau \sqrt{(\upsilon^2 - \tau^2)(\bar{\upsilon}^2 - \tau^2)}).$$

(6.20)

Next, from (6.17), we can have

$$\langle z_0, f \rangle = \left\{ 1 - \frac{1}{2\pi y_2(\tau)} \int_{|y|<|\tau|} \mathfrak{Q}(x + i0) \frac{f(x)}{x - \tau} \, dx \right\} f(\tau)$$

$$+ \frac{1}{2\pi y_2(\tau)} \int_{|y|<|\tau|} \mathfrak{Q}(x + i0) \frac{f(x)}{x - \tau} \, dx,$$

(6.21)

where $P$ means principal value of the integral.

But from (6.7), the following limit relationship holds:

$$\Re Q(x + i0) = \frac{1}{\pi} P \int_{|y|<|x|<\pi} \frac{\mathfrak{Q}(t + i0)}{t - x} \, dt, \quad x \in \mathbb{R}.$$ 

(6.22)

With (6.19), this gives

$$\frac{1}{\pi} P \int_{y < |t| < \pi} \frac{\mathfrak{Q}(t + i0)}{t - x} \, dt = -W(x), \quad \upsilon < |x| < \bar{\upsilon}.$$ 

(6.23)

Consequently, (6.21) becomes

$$\langle z_0, f \rangle = -\frac{1}{2} i y_2^{-1}(\tau) \text{sgn} \tau \sqrt{(\upsilon^2 - \tau^2)(\bar{\upsilon}^2 - \tau^2)} f(\tau)$$

$$+ \frac{1}{2\pi y_2(\tau)} \int_{|y|<|\tau|<\pi} \mathfrak{Q}(x + i0) \frac{f(x)}{x - \tau} \, dx.$$ 

(6.24)

(b) $|\tau| < \upsilon$. From (6.13), we still have (6.17), where here

$$y_2(\tau) = \frac{1}{2} \left( \sqrt{(\upsilon^2 - \tau^2)(\bar{\upsilon}^2 - \tau^2)} - W(\tau) \right).$$ 

(6.25)
Taking (6.19) and (6.22) into account, we infer that

\[
\frac{1}{\pi} P \int_{|t|<\overline{u}} \frac{\mathcal{Q}(t+i0)}{t-\tau} \, dt = -\left( \sqrt{(\overline{\nu}^2 - \tau^2)} (\overline{\nu}^2 - \tau^2) + W(\tau) \right). 
\]  

(6.26)

Thus, we obtain

\[
\langle z_0, f \rangle = \gamma_2^{-1}(\nu) \sqrt{(\nu^2 - \tau^2)(\overline{\nu}^2 - \tau^2)} f(\tau) + \frac{1}{2\pi \gamma_2(\tau)} \int_{|x|<\overline{\nu}} \frac{\mathcal{Q}(x+i0)}{x-\tau} f(x) \, dx.
\]  

(6.27)

These results are summarized in the following proposition.

**Proposition 6.1.** Suppose either \(0 \leq \nu < 1\) or \(\nu > 1\). Let \(\nu := \min(1, \nu)\) and \(\overline{\nu} := \max(1, \nu)\).

Then the form \(z_0\) possesses the following integral representation:

1. For \(\tau \in \mathbb{C} - \overline{\nu} + \mathbb{C}\),

\[
\langle z_0, f \rangle = -\gamma_2^{-1}(\nu) \sqrt{(\nu^2 - \tau^2) (\overline{\nu}^2 - \tau^2)} f(\tau) \]

\[+ \frac{1}{2\pi \gamma_2(\tau)} \int_{|x|<\overline{\nu}} \frac{\mathcal{Q}(x+i0)}{x-\tau} f(x) \, dx; \]

(6.28)

2. For \(\nu < |\tau| < \overline{\nu}\),

\[
\langle z_0, f \rangle = -\frac{1}{2} i \gamma_2^{-1}(\nu) \text{sgn} \tau \sqrt{(\nu^2 - \tau^2)(\overline{\nu}^2 - \tau^2)} f(\tau) \]

\[+ \frac{1}{2\pi \gamma_2(\tau)} P \int_{|x|<\overline{\nu}} \frac{\mathcal{Q}(x+i0)}{x-\tau} f(x) \, dx; \]

(6.29)

3. For \(|\tau| \leq \nu\),

\[
\langle z_0, f \rangle = \gamma_2^{-1}(\nu) \sqrt{(\nu^2 - \tau^2)(\overline{\nu}^2 - \tau^2)} f(\tau) \]

\[+ \frac{1}{2\pi \gamma_2(\tau)} \int_{|x|<\overline{\nu}} \frac{\mathcal{Q}(x+i0)}{x-\tau} f(x) \, dx.
\]  

(6.30)

**Remark 6.2.** In the last case \(|\tau| \leq \nu\), the form \(z_0\) is positive definite since \(\gamma_1(\tau) > 0\) and \(\gamma_2(\tau) > 0\).

Regarding the moments, from (6.1), we easily obtain

\[
(z_0(\tau, \nu, +1))_{2n} = \sum_{\mu=0}^{n} \tau^{2(n-\mu)} d_{\mu}, \quad n \geq 0, \]

(6.31)

\[
(z_0(\tau, \nu, +1))_{2n+1} = \tau (z_0(\tau, \nu, +1))_{2n}, \quad n \geq 0,
\]
The second-order self-associated orthogonal sequences

where

\[ d_0 = 1, \quad d_n = -\frac{1}{2} y_2^{-1} c_{n+1}, \quad n \geq 1, \]

\[ c_n = \frac{1}{4\pi} \sum_{m+k=n} \frac{\Gamma(m-1/2)}{m!} \frac{\Gamma(k-1/2)}{k!} \nu^{2k}, \quad n \geq 0. \]  

(6.32)

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References


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