A GENERALIZATION OF A THEOREM
BY CHEO AND YIEN CONCERNING DIGITAL SUMS

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ABSTRACT. For a non-negative integer n, let s(n) denote the digital sum of n. Cheo
and Yien proved that for a positive integer x, the sum of the terms of the sequence
\{s(n) : n = 0, 1, 2, \ldots, (x-1)\}
is (4.5)x\log x + O(x). In this paper we let k be a positive integer and determine that
the sum of the sequence
\{s(kn) : n = 0, 1, 2, \ldots, (x-1)\}
is also (4.5)x\log x + O(x). The constant implicit in the big-oh notation is dependent
on k.

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INTRODUCTION.

In Cheo and Yien [1], it was proven that for a positive integer x,
\[ \sum_{n=0}^{x-1} s(n) = (4.5)x\log x + O(x) \] (1.1)
where s(n) denotes the digital sum of n. Here, we will show that, in fact, for any
positive integer k,
\[ \sum_{n=0}^{x-1} s(kn) = (4.5)x\log x + O(x) \] (1.2)
where the constant implicit in the big-oh notation is dependent on k.

The following notation will be used to facilitate the proof of (1.2). For integers
x and y,
\[ x \mod y \] (1.3)
will be the remainder when x is divided by y and, as usual, square brackets will denote
the integral part operator. In addition, for non-negative integers m, i, and j we let
\[ [m]^i = m \mod 10^i, \] (1.4)
\[ [m]_i = [m/10^i], \] (1.5)
and
\[
[m]^j_i = \left\lfloor [m]^j \right\rfloor_i
\]
for \( i < j \).

Thus, the \( j \) right-most digits of \( m \) are given by (1.4) and the number determined by dropping the \( i \) right-most digits of \( m \) is given by (1.5). Therefore, the number determined from the \( j \)th right-most digit of \( m \) to the \((i + 1)\)st right-most digit of \( m \) is given by (1.6).

2. A PROOF OF (1.2) WHEN \( k \) AND 10 ARE RELATIVE PRIME.

Let \( (k, 10) = 1 \), \( x \) be a positive integer, and \( L = \lfloor \log x \rfloor \). Then
\[
\sum_{n=0}^{x-1} s([kn]) = \sum_{n=0}^{x-1} s([kn]^L) + \sum_{n=0}^{x-1} s([kn]_L) \quad (2.1)
\]
\[
= \sum_{n=0}^{x-1} s([kn]^L) + O(x) \quad (2.2)
\]
This follows since for non-negative integers \( L \) and \( m \),
\[
m = [m]^L + 10^L [m]_L \quad (2.3)
\]
and so
\[
s(m) = s([m]^L) + s([m]_L) \quad (2.4)
\]
Also, since each \( s([kn]_L) \) is bounded by a constant (dependent on \( k \)), we have that the second term of (2.1) is \( O(x) \).

Next, for \( i = 0, 1, 2, \ldots, L \) define
\[
x_i = [x]_{L+1-i} 10^{L+1-i} \quad (2.5)
\]
Then,
\[
\sum_{n=0}^{x-1} s([kn]^L) = \sum_{n=x_1}^{x_1-1} s([kn]^L) + \sum_{n=x_1}^{x_1-1} s([kn]_L) \quad (2.6)
\]
\[
= \sum_{n=x_1}^{x_1-1} s([kn]^L) + \sum_{n=x_1}^{x_1-1} s([kn]_L) + \sum_{n=x_1}^{x_2} s([kn]_L-1) + \sum_{n=x_1}^{x_2} s([kn]_L-2) \quad (2.7)
\]
In the same way,
\[
\sum_{n=x_1}^{x_2} s([kn]_L-1) = \sum_{n=x_2}^{x_2-1} s([kn]_L-1) + \sum_{n=x_2}^{x_2-1} s([kn]_L-2) + \sum_{n=x_2}^{x_3} s([kn]_L-3) \quad (2.7)
\]
Continuing in this manner and combining terms, we have
\[
\sum_{n=0}^{x-1} s([kn]_L) = \sum_{i=0}^{L} \sum_{n=x_i}^{x_i-1} s([kn]_L^{L+1-i}) + \sum_{i=1}^{L} \sum_{n=x_i}^{x_i-1} s([kn]_L^{L+1-i}) \quad (2.8)
\]
Since
\[ s([kn]_{L+1-i}) \quad \text{(2.9)} \]
is a decimal digit and
\[ x - x_i = [x]^{L+1-i} \leq 10^{L+1-i} \quad \text{(2.10)} \]
for each \( i \), it follows that
\[ \sum_{i=1}^{L} \frac{x - 1}{n} \quad s([kn]_{L+1-i}) = O(x) \quad \text{(2.11)} \]

To determine the value of the first term of (2.8), we need the following lemma. Its proof is straightforward and will not be given.

**Lemma 2.** Let \( d \) and \( i \) be non-negative integers. Then for \( (k,10) = 1 \),
\[ \{[kn]^{i} \colon n = d, d+1, \ldots, d+10^i-1\} = \{n \colon n = 0, 1, \ldots, 10^i-1\}. \quad \text{(2.12)} \]

By this lemma and the fact that
\[ x - x_i = [x]^{L+2-i} \quad 10^{L+1-i} \quad \text{(2.13)} \]
it follows that
\[ \sum_{i=1}^{L} \frac{x - 1}{n} \quad s([kn]_{L+1-i}) = (\lfloor x \rfloor_{L+1-i}) \quad 10^{L+1-i} \quad \text{(2.14)} \]
for each \( i \).

Now since
\[ \sum_{n=0}^{10^{L+1-i}-1} s(n) = 4.5(L + 1 - i)10^{L+1-i} \quad \text{(2.15)} \]
by [2], we have that
\[ \sum_{i=1}^{L} \frac{x - 1}{n} \quad s([kn]_{L+1-i}) = (4.5)x\log x + O(x) \quad \text{(2.16)} \]
Using (2.16) and (2.11) in (2.8), by (2.2) we have the expression given in (1.2). The constant implicit in the big-oh notation is dependent on \( k \) with \( k \) and 10 relatively prime.

3. **Conclusion.**

For any positive integer \( k \), there exists non-negative integers \( a, b, \) and \( r \) such that \( k = 2^a5^b r \) with \( (r,10) = 1 \). Note that if \( k = r \), then we have (1.2). However, by use of the following generalization to Lemma 2, and some technical modifications, it can be shown that the restriction that \( k \) and 10 be relatively prime can be removed in the derivation of (2.1). That is,
\[ \sum_{n=0}^{x - 1} s(n) = (4.5)x\log x + O(x) \quad \text{(3.1)} \]
for any positive integer \( k \).

**Lemma 3.** Let \( k = 2^a5^b r \) with \( (r,10) = 1 \) and \( i = \max \{a,b\} \). Then for any non-
negative integer $d$,

$$([kn]^d : n = d, d+1, d+2, \ldots, d + (10^d/2^{a5b}) - 1)$$

$$= \{2^{a+b} n : n = 0, 1, 2, \ldots, (10^d/2^{a5b}) - 1\}.$$  \hspace{3cm} (3.2)

Finally, based on the above techniques, it is strongly conjectured that for any positive integers $k_1$ and $k_2$, it again follows that

$$\sum_{n=0}^{x-1} s(k_1 n + k_2) = (4.5)x\log x + O(x).$$  \hspace{3cm} (3.3)

REFERENCES
