ON THE NON-EXISTENCE OF SOME INTERPOLATORY POLYNOMIALS

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ABSTRACT. Here we prove that if \( x_k, k = 1,2,\ldots,n + 2 \) are the zeros of \((1 - x^2)T_n(x)\) where \( T_n(x) \) is the Chebyshev polynomial of first kind of degree \( n \), \( \alpha_j, \beta_j, j = 1,2,\ldots,n + 2 \) and \( \gamma_j, j = 2,3,\ldots,n + 1 \) are any real numbers there does not exist a unique polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n + 3 \) satisfying the conditions:

\[
Q_{3n+3}(x_j) = \alpha_j, \quad Q_{3n+3}'(x_j) = \beta_j, \quad j = 1,2,\ldots,n + 2 \quad \text{and} \quad Q_{3n+3}(x_j) = \gamma_j, \quad j = 2,3,\ldots,n + 1.
\]

Similar result is also obtained by choosing the roots of \((1 - x^2)P_n(x)\) as the nodes of interpolation where \( P_n(x) \) is the Legendre polynomial of degree \( n \).

KEY WORDS AND PHRASES. Roots, interpolatory polynomials, non-existence, nodes.

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1. INTRODUCTION.

In [1] R.B. Saxena considered an interesting problem of (0,1,3) interpolation by taking the roots of \((1 - x^2)P_{n-2}(x)\), where \( P_{n-2}(x) \) is the Legendre polynomial of degree \( n - 2 \), as the nodes of interpolation. By (0,1,3) interpolation, Saxena meant that for the collections \( \{\alpha_j\}_{j=1}^{n}, \{\beta_j\}_{j=2}^{n-1}, \) and \( \{\gamma_j\}_{j=1}^{n} \) of real numbers and the zeros \( x_j \) of \((1 - x^2)P_{n-2}(x)\) arranged so that

\[-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1\]

a polynomial \( R_n(x) \) of degree \( \leq 3n - 3 \) can be constructed so that

\[R_n(x_j) = \alpha_j; \quad j = 1,2,\ldots,n,\]

\[R_n'(x_j) = \beta_j; \quad j = 2,3,\ldots,n - 1,\]

and

\[R_n(x_j) = \gamma_j; \quad j = 1,2,\ldots,n.\]

Saxena proved that such a polynomial exists uniquely if \( n \) is even and for \( n \) odd there does not exist a unique polynomial \( R_n(x) \) satisfying the above conditions.

Later Varma [2] obtained the following result in this direction:

THEOREM 1 (VARMA). Given a positive integer \( n \) and real numbers \( \alpha_k(k = 1,2,\ldots,n + 2), \beta_k, \gamma_k(k = 2,3,\ldots,n + 1) \) there is, in general no polynomial \( F_{3n+1}(x) \) of degree \( \leq 3n + 1 \) such that \( F_{3n+1}(x_k) = \alpha_k; k = 1,2,\ldots,n + 2, F_{3n+1}'(x_k) = \beta_k; \)
k = 2, 3, ..., n + 1 and \( F_{3n+1}^{(r)}(x_k) = Y_k \); k = 2, 3, ..., n + 1 provided \( x_k \)'s are the zeros of \((1 - x^2)T_n(x)\) where \( T_n(x) \) is Tchebycheff polynomial of first kind and if there exists such a polynomial then there is an infinity of them.

2. MAIN RESULTS.

In connection with the above results we shall prove the following.

THEOREM 2. For any positive integer \( n \), with \( 1 = \xi_1 > \xi_2 > \ldots > \xi_{n+1} > \xi_{n+2} = -1 \) the zeros of \((1 - x^2)P_n(x)\) where \( P_n(x) \) is the Legendre polynomial of degree \( n \), there is in general no polynomial \( R_{3n+1}(x) \) of degree \( 3n + 1 \) such that, for arbitrary real numbers \( \{a_j\}_{n+2}^{j=1}, \{\beta_j\}_{n+1}^{j=2} \) and \( \{\gamma_j\}_{n+1}^{j=2} \) the conditions:

\[
R_{3n+1}(\xi_j) = a_j; j = 1, 2, \ldots, n + 1, n + 2, \tag{2.1}
\]

\[
R_{3n+1}(\xi_j) = \beta_j; j = 2, 3, \ldots, n + 1 \tag{2.2}
\]

and

\[
R_{3n+1}''(\xi_j) = \gamma_j; j = 2, 3, \ldots, n + 1 \tag{2.3}
\]

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

We also prove the following result for Tchebycheff nodes:

THEOREM 3. For any positive integer \( n \), with \( 1 = x_1 > x_2 > \ldots > x_n > x_{n+1} > x_{n+2} = -1 \) the zeros of \( \omega_n(x) = (1 - x^2)T_n(x) \), there is in general no polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n + 3 \) such that for arbitrary real numbers \( \{a_j\}_{n+2}^{j=1}, \{\beta_j\}_{n+1}^{j=2} \) and \( \{\gamma_j\}_{n+1}^{j=2} \) the conditions:

\[
Q_{3n+3}(x_j) = a_j; j = 1, 2, \ldots, n + 1, n + 2, \tag{2.4}
\]

\[
Q_{3n+3}(x_j) = \beta_j; j = 1, 2, \ldots, n + 1, n + 2 \tag{2.5}
\]

and

\[
Q_{3n+3}''(x_j) = \gamma_j; j = 2, 3, \ldots, n + 1 \tag{2.6}
\]

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

REMARK 1. The comparison of our Theorem 2 with the above mentioned result of Saxena shows that if we do not prescribe the third derivative at \( \pm 1 \) then there does not exist a unique polynomial regardless whether \( n \) is even or odd. In an earlier work [3] we have shown that along with the conditions (2.1), (2.2) and (2.3) if we also prescribe the first derivative at \( \pm 1 \) a unique polynomial of degree \( \leq 3n + 3 \) still does not exist. It is also evident from Theorem 3 that even if we prescribe the first derivative at \( \pm 1 \) a unique polynomial of degree \( \leq 3n + 3 \) does not exist although the nodes of interpolation are different from that of [3].

REMARK 2. We shall give here the proof of Theorem 3 only. The proof of Theorem 2 can be obtained along the same lines.

PROOF OF THEOREM 3. We will show that if all of

\[
\alpha_j = 0; j = 1, 2, \ldots, n + 1, n + 2, \tag{2.7}
\]

\[
\beta_j = 0; j = 1, 2, \ldots, n + 1, n + 2,
\]

\[
\gamma_j = 0; j = 2, 3, \ldots, n + 1
\]


then there exists a polynomial $Q_{3n+3}(x)$ of degree $\leq 3n + 3$ which is not identically zero, but satisfies (2.4), (2.5) and (2.6). The desired result then follows immediately from the theory of linear equations. From the definition of $\omega_n(x)$ and conditions (2.4), (2.5) and (2.6), together with the requirements (2.7), it is clear that the desired polynomial must be of the form

$$Q_{3n+3}(x) = (1 - x^2)^2 T_n^2(x) k_{n-1}(x)$$

(2.8)

where $k_{n-1}(x)$ is an unknown polynomial of degree $\leq n - 1$. Since we have also required $Q_{3n+3}(x_j) = 0$ for $j = 2, 3, \ldots, n + 1$, simple calculation provides

$$(1 - x^2) k'_{n-1}(x) - 3x k_{n-1}(x) = c T_n(x)$$

(2.9)

for unknown real constant $c$. Letting $x = \cos \theta$ and

$$k_{n-1}(x) = \sum_{k=0}^{n-1} a_k \cos k\theta$$

we obtain

$$k'_{n-1}(x) = \sum_{k=1}^{n-1} a_k \sin k\theta \sin \theta.$$  

Thus (2.9) becomes

$$c \cos n\theta = \sum_{k=0}^{n-1} a_k [k \sin k\theta \sin \theta - 3 \cos k\theta \cos \theta].$$

From this, we obtain on simplification

$$2c \cos n\theta = \sum_{k=0}^{n-1} a_k [(k - 3) \cos (k - 1)\theta - (k + 3) \cos (k + 1)\theta],$$

from which, by collecting the coefficients of $\cos k\theta$, for $k = 0, 1, \ldots, n$, we may write

$$-2a_1 = (6a_0 + a_2) \cos \theta - 4a_1 \cos 2\theta$$

$$+ \sum_{k=3}^{n-2} \{(k - 2)a_{k+1} - (k + 2)a_{k-1}\} \cos k\theta$$

$$-(n + 1)a_{n-2} \cos (n - 1)\theta - (n + 2)a_{n-1} \cos n\theta$$

$$= 2c \cos n\theta.$$

This, in turn, leads to the following system of equations

$$-2a_1 = 0$$

$$-(6a_0 + a_2) = 0,$$

$$-4a_1 = 0,$$

$$(k - 2)a_{k+1} - (k + 2)a_{k-1} = 0; \ k = 3, 4, \ldots, n - 2,$$

$$-(n + 1)a_{n-2} = 0,$$

$$-(n + 2)a_{n-1} = 2c.$$

If $n$ is even, then

$$a_0 = a_2 = a_4 = \ldots = a_{n-2} = 0; \ a_1 = 0$$
but
\[ a_{n-1-2j} = \frac{-2c}{n-2} \prod_{k=0}^{j} \left( \frac{n-2-2k}{n+2-2k} \right); \text{ for } j = 0, 1, \ldots, (n-4)/2 \]
is not necessarily zero.

If \( n \) is odd, then
\[ a_1 = a_3 = a_5 = \ldots = a_{n-2} = 0, \]
while
\[ a_{2j} = \frac{-2c}{n-2} \prod_{k=j}^{(n-1)/2} \frac{2k-1}{2k+3}; j = 1, 2, \ldots, (n-1)/2 \]
with the special case
\[ a_0 = a_2/6 \]
which are not necessarily zero. Hence regardless whether \( n \) is even or odd, in general, there does not exist a unique polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n+3 \) satisfying (2.4), (2.5) and (2.6) and there are infinitely many if they exist.

This completes the proof of Theorem 3. For a complete history on lacunary interpolation we refer to a paper by J. Balázs [4].

REFERENCES