ON COLLINEATION GROUPS OF TRANSLATION PLANES OF ORDER \( q^4 \)

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ABSTRACT. Let \( P \) be an affine translation plane of order \( q^4 \) admitting a nonsolvable group \( G \) in its translation complement. If \( G \) fixes more than \( q^4 + 1 \) slopes, the structure of \( G \) is determined. In particular, if \( G \) is simple then \( q \) is even and \( G \cong L_2(2^s) \) for some integer \( s \) at least 2.

KEY WORDS AND PHRASES. Translation planes, nonsolvable groups.

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1. INTRODUCTION.

Let \( \Pi \) be a translation plane of square order \( p^{2r} \). If \( \Pi \) admits a collineation group isomorphic to \( SL(2,p^t) \) and the Sylow \( p \)-subgroups are planar, then usually (in the known cases) the group fixes \( p^{r+1} \) components (or slopes). Generally, however, simply knowing that a group fixes a number of slopes says essentially nothing concerning the structure of the group. However, for planes of order \( q^4 \), we can make some progress. That is, in this note our objective is to prove the following.

THEOREM A. Let \( \Pi \) be an affine translation plane of order \( q^4 \) admitting a nonsolvable group \( G \) in its translation complement. Suppose \( G \) fixes more than \( q^4 + 1 \) slopes.

(1) If \( q \) is odd then \( 8 \mid |G| \) and the 2-rank of \( G \leq 2 \). Furthermore, \( G \) always contains the kern involution.

(2) If \( q \) is even then \( G \) contains a normal subgroup \( N \) such that \( N \cong L_2(2^s) \), for some \( s \), and \( G/N \) is of odd order. Now the Sylow 2-subgroups fix Baer subplanes elementwise. Furthermore, the Baer subplanes share the same points at infinity and so \( N \) fixes exactly \( q^{2s+1} \) slopes.

COROLLARY 1. If \( G \) is simple then \( q \) is even and \( G \cong L_2(2^s) \) for some integer \( s \geq 2 \).

PROOF. When \( q \) is odd the kern involution is central in \( G \) and so \( G \) is not simple.
REMARKS. (1) When \( q \) is odd it is possible to find \( G \)'s with 2-rank one and 2-rank two such that they satisfy the hypothesis of Theorem A. For example, \( G_1 = \text{SL}(2,q) \), acting on Hall planes of order \( q^4 \), has 2-rank one; if we now choose \( G_2 = \langle g_1, a \rangle \), where \( a \) is any Baer involution, we get a \( G \) with 2-rank two.

(2) Both the theorem and its corollary cease to be unconditionally true if we allow \( G \) to fix "at least \( q+1 \) slopes", instead of "more than \( q+1 \) slopes": the only known counterexamples seem to be the Lorimer-Rahilly planes \([1]\) and its transpose, the Johnson-Walker plane \([2]\).

The following well known consequences of Foulser \([3]\) will be used on several occasions in the proof of Theorem A.

RESULT 0. Suppose \( B \) is a collineation group of an affine translation plane of order \( p^{2r} \) that fixes a Baer subplane elementwise. Then

(i) \( B \) is solvable;
(ii) the Sylow \( p \)-subgroups of \( B \) are elementary abelian; and
(iii) the Hall \( p' \) subgroups of \( B \) are cyclic.

2. PROOF OF THEOREM A.

We begin by dealing with the case when \( q \) is odd. The first step is folklore and corresponds to Ostrom's ideas in \([4]\).

**Lemma 1.** If \( q \) is odd then any Klein 4-group in \( G \) must contain the unique kern involution of \( \pi \); we shall always denote this involution by \( \hat{\pi} \).

**Proof.** Let \( K = \{1, \alpha, \beta, \alpha \beta \} \) be a Klein group in \( G \) and suppose, if possible, \( \hat{\pi} \notin K \). For any involution \( x \) in \( K \) write \( \pi_x \) for its fixed Baer subplane. Now we claim \( \pi_x \cap \pi_y \) cannot be a subplane of \( \pi \) if \( x, y \) are distinct involutions in \( K \). If \( \pi_x = \pi_y \), we have a Klein group fixing elementwise a Baer subplane of \( \pi \), contrary to Result 0. So \( \pi_x \cap \pi_y \) is a fourth root subplane of \( \pi \) and now we contradict the assumption that \( G \) fixes more than \( q+1 \) slopes. Thus \( xy \) acts like \(-1\) on the fixed components of \( G \), in the spread associated with \( \pi \); i.e., \( xy \) is the required involution.

**Lemma 2.** If \( q \) is odd then \( |G| \) is divisible by 8.

**Proof.** If 2 exactly divides \( |G| \) then, by Burnside's theorem, \( G \) has a normal 2-complement \([5, 6.2.11]\) and we contradict the assumption that \( G \) is nonsolvable. For the same reason the Sylow 2-subgroups of \( G \) cannot be cyclic of order 4. Hence \( 4 \nmid |G| \) only if the Sylow 2-subgroups are Klein groups. So by Lemma 1, \( G \) contains the kern involution \( \hat{\pi} \) and \( G/\langle \hat{\pi} \rangle \) is solvable. Thus we contradict the nonsolvability of \( G \) when \( 8 \nmid |G| \). The result follows.

**Lemma 3.** Suppose \( q \) is odd. Then \( G \) cannot contain an elementary abelian 2-group of order 8.

**Proof.** If \( S \) is an elementary abelian subgroup of \( G \), whose order is 8, we may write

\[
S = \{1, \alpha, \beta, \alpha \beta, \gamma, \alpha \gamma, \beta \gamma, \alpha \beta \gamma\}
\] (2.1)
and assume that
\[ L = \{1, \alpha, \beta, \alpha \beta \}, \quad M = \{1, \alpha, \gamma, \alpha \gamma \} \]  
are distinct subgroups of order 4. But by Lemma 1, both \( L \) and \( M \) contain the kernel involution \( \hat{i} \). Hence \( \alpha = \hat{i} \). Interchanging the role of \( \alpha \) and \( \beta \), we find that \( \beta \) is also \( \hat{i} \). The lemma follows, since we have contradicted the assumption that \( |S| = 8 \).

**Lemma 4.** \( G \) contains \( \hat{\pi} \), the kernel involution of \( \pi \).

**Proof.** Let \( S \) denote a Sylow 2-subgroup of \( G \). So \( |S| \geq 8 \) (Lemma 2) and non-cyclic, because \( G \) is nonsolvable. Now Lemma 1 applies unless the 2-rank of \( S \) is one. Thus \( S \) is the generalized quaternion group
\[ \langle x, y \mid x^{2n} = 1, y^2 = x^{2n-1}, y^{-1}xy = x^{-1} \text{ for } n \geq 2 \rangle \]  
and so contains \( Q = \langle x^{2n-2}, y \rangle \), the quaternion group of order 8.

Now let \( \alpha \) denote the unique involution in \( Q \) and, to get a contradiction, assume \( \alpha \) is a Baer involution with fixed plane \( \pi_\alpha \). Now \( Q \) leaves \( \pi_\alpha \) invariant but does not fix it elementwise because of Result 0. Moreover, no element of \( Q \) can induce a Baer involution on \( \pi_\alpha \) because \( Q \) fixes \( q+1 \) slopes of \( \pi_\alpha \). Thus the restriction map
\[ \rho : Q \longrightarrow Q \mid \pi_\alpha \]  
has as its image \( \langle \beta \rangle \), where \( \beta \) is the kernel involution of \( \pi_\alpha \). So \( \ker \rho \) is clearly a noncyclic group \( \Sigma \) of order 4, contrary to Result 0.

The lemmas proved so far add up to Theorem A, Part (i). To deal with the case when \( q \) is even we need the following version of a theorem of Johnson [6], deduced from Hering [7].

**Result 5.** Suppose \( \psi \) is an affine translation plane of even order admitting a nonsolvable group \( H \) in its translation complement. Assume a Sylow 2-subgroup of \( H \) fixes a Baer subplane elementwise. Then \( H \) contains a normal subgroup \( N \) such that \( H/N \) is of odd order and \( N \cong L_2(2^s) \) for some integer \( s \).

**Proof.** Use Johnson's argument in [6, Theorem 2.3].

**Lemma 6.** If \( q \) is even then the Sylow 2-subgroups of \( G \) fix Baer subplanes of \( \pi \) elementwise.

**Proof.** Let \( S \) be a Sylow 2-subgroup of \( G \) and note that elements of \( \pi \) fixed by \( S \) form a subplane \( \pi_S \), because \( S \) fixes many slopes. To get a contradiction we assume \( \pi_S \) is not a Baer subplane of \( \pi \). Now let \( \alpha \) be any involution in the center of \( S \) and let \( \pi_\alpha \) be its fixed Baer subplane. Then we have a chain of planes \( \pi \supseteq \pi_\alpha \supseteq \pi_S \). Hence the order of \( \pi_S \) is \( q+1 \) and we contradict the assumption that \( G \) fixes more than \( q+1 \) slopes. The result follows.

Now by Result 5 and Lemma 6 we immediately have

**Lemma 7.** Suppose \( q \) is even. Then the Sylow 2-subgroups of \( G \) fix Baer subplanes elementwise and generate of a subgroup \( N \cong L_2(2^s) \) where \( 2^s \| |G| \).
To complete the proof of Theorem A we restrict ourselves from now on to the situation described in Lemma 7.

**Lemma 8.** \(N\) fixes a unique affine point 0.

**Proof.** Suppose \(N\), which is in the translation complement of \(\pi\), fixes a second affine point of \(\pi\). Then \(N\) is a planar group of \(\pi\) of order \(> q\). Now by Lemma 7 we have a Baer chain \(\pi \supset \pi_S \supset \pi_N\). If \(\pi_N \neq \pi_S\), we have the same contradiction as in Lemma 6; otherwise we have a nonsolvable group fixing a Baer subplane element-wise, contrary to Result 0.

**Lemma 9.** The only affine point fixed by distinct Sylow 2-subgroups of \(N\) is 0, the unique point fixed by \(N\).

**Proof.** \(N\) is generated by any two of its Sylow 2-subgroups, so we contradict Lemma 8 unless the lemma is valid.

For \(2^8 \neq 4\), Lemma 9, when combined with Foulser and Johnson [8, Proposition 3.4], shows than \(N\) fixes \(q^2 + 1\) slopes. From Johnson [9, Theorem 2.1], the same is true if \(2^8 = 4\) as \(N\) fixes at least 3 slopes. This proves Part (2) of Theorem A.

**References**

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