ABSTRACT. Let \( A \) be a nonvoid countable subset of the unit interval \([0,1]\) and let \( B \) be an \( F_\sigma \)-subset of \([0,1]\) disjoint from \( A \). Then there exists a derivative \( f \) on \([0,1]\) such that \( f = 0 \) on \( A \), \( f > 0 \) on \( B \), and such that the extended real valued function \( 1/f \) is also a derivative.

KEY WORDS AND PHRASES. Derivative, primitive, Lebesgue summable, knot point.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 26A24

In this note, we construct a derivative \( f \) such that \( 1/f \) is also a derivative, and \( f \) and \( 1/f \) have some curious properties mentioned in [1] and [2]. (By an \( F_\sigma \)-set in the real line, we mean the union of countably many closed subsets of \( \mathbb{R} \).) We prove

THEOREM 1. Let \( A \) be a nonvoid countable subset of \([0,1]\) and let \( B \) be an \( F_\sigma \)-subset of \([0,1]\) disjoint from \( A \). Then there exists a measurable function \( f \) on \([0,1]\) such that

1. \( f = 0 \) on \( A \), \( f > 0 \) on \( B \), \( 0 \leq f \leq 1 \) on \([0,1]\) and
2. \( f \) is everywhere the derivative of its primitive,
3. \( 1/f \) is Lebesgue summable on \([0,1]\),
4. \( 1/f \) is everywhere the derivative of its primitive.

Here we let \( m = 1/0 \).

When \( m(B) = 1 \) and \( A \) is dense, we will obtain a simple example of a derivative that vanishes on a dense set of measure 0.

From [2] we infer that there exists a derivative \( f \) vanishing on \( A \) and positive on \( B \). From [1] we infer that there exists a derivative \( g \) infinite on \( A \) and finite on \( B \). However, Theorem 1 provides a simultaneous solution to both of these problems. To prove Theorem 1 we will employ some of the methods used in [3].

Finally, we use these methods to construct a concrete example of functions \( g_1 \) and \( g_2 \) that have finite or infinite derivatives at each point, such that the Dini derivatives of their difference, \( g_1 - g_2 \), satisfy certain pathological properties.

In all that follows, let \( (n(i))_{i=1}^\infty \) denote the sequence of integers \( 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, ... \).

Proof of Theorem 1. Let \((a_i)_{i=1}^\infty \) be a sequence of points in \( A \) such that each point of \( A \) occurs at least once in the sequence. (Here we do not exclude the possibility that \( A \) is a finite set.) We assume, without loss of generality, that \( B \) is nonvoid.
Let $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \cdots$ be an expanding sequence of closed sets such that $B = \bigcup_{i=1}^{\infty} B_i$. (Here we do not exclude the possibility that $B$ is a closed set.) Let $u_i$ denote the distance from the point $a \in A_i$ to the set $B_i$. As in [3], we put $\theta(x) = (1 + |x|)^{-1}$.

For each index $j$, put
\[
g_j(x) = 1 + \sum_{i=1}^{j} \theta(2^i u_i^{-1}(x-a_{n(i)})),
\]
\[
g(x) = 1 + \sum_{i=1}^{\infty} \theta(2^i u_i^{-1}(x-a_{n(i)})),
\]
\[
f_j(x) = 1/g_j(x), \quad f(x) = 1/g(x).
\]

Here we let $0 = 1/\infty$. Then $g(a) = \infty$ for $a \in A$, because there are infinitely many indices $i$ for which $a \neq a_{n(i)}$. On the other hand, $g(b) < \infty$ for $b \in B$; note that if $b \in B_k$, then
\[
\theta(2^k u_k^{-1}(b-a_{n(k)})) \leq \theta(2^k) < 2^{-1/2},
\]
\[
\sum_{i=k}^{\infty} \theta(2^i u_i^{-1}(b-a_{n(i)})) \leq \sum_{i=k}^{\infty} 2^{-i/2} < \infty.
\]

We infer from Lemma 4 of [3], that $g$ is Lebesgue summable on $[0,1]$. Note also that
\[
g(x) - g_j(x) = g(x)g_j(x)(f_j(x) - f(x)) > 0,
\]
and since $g > 1$, $g_j > 1$, it follows that $g - g_j > f_j - f > 0$.

Now choose any $x$ with $g(x) < \infty$. By Lemma 4 of [3], we have
\[
limit_{h \to 0} h^{-1} \int_{x}^{x+h} g(t)dt = g(x).
\]

Take any $\epsilon > 0$. Select an index $j$ so large that $f_j(x) - f(x) < g(x) - g_j(x) < \epsilon$. Since $f_j$ and $g_j$ are continuous, when $|h|$ is small enough we have
\[
|h^{-1} \int_{x}^{x+h} g(t) dt - g(x)| < \epsilon,
\]
\[
|h^{-1} \int_{x}^{x+h} g_j(t) dt - g_j(x)| < \epsilon,
\]
\[
|h^{-1} \int_{x}^{x+h} f_j(t) dt - f_j(x)| < \epsilon.
\]

For such $j$ and $h$ we obtain
\[
h^{-1} \int_{x}^{x+h} (g(t) - g_j(t))dt \leq g(x) - g_j(x) + |h^{-1} \int_{x}^{x+h} g(t) dt - g(x)|
\]
\[
+ |h^{-1} \int_{x}^{x+h} g_j(t) dt - g_j(x)| < 3\epsilon.
\]

From $0 < f_j - f < g - g_j$ we obtain
\[
|h^{-1} \int_{x}^{x+h} f(t) dt - f(x)| \leq |h^{-1} \int_{x}^{x+h} f_j(t) dt - f_j(x)| + f_j(x) - f(x)
\]
\[
+ h^{-1} \int_{x}^{x+h} (f_j(t) - f(t)) dt
\]
\[
\leq 2\epsilon + h^{-1} \int_{x}^{x+h} (f_j(t) - f(t)) dt
\]
\[
\leq 2\epsilon + h^{-1} \int_{x}^{x+h} (g(t) - g_j(t)) dt < 5\epsilon.
\]

It follows that $\limit_{h \to 0} h^{-1} \int_{x}^{x+h} f(t) dt = f(x)$.
Choose any \(x\) with \(g(x) = \infty\). Take any \(N > 0\). Select \(j\) so large that \(g_j(x) > N\).

Since \(g_j\) is continuous, there is a \(d > 0\) such that \(|t - x| < d\) implies \(g_j(t) > N\). For such \(t\), \(g(t) > g_j(t) > N\) and \(f(t) < f_j(t) < N^{-1}\). It follows that for \(|h| < d\),

\[
\int_{x}^{x+h} g(t) \, dt > N, \quad 0 < \int_{x}^{x+h} f(t) \, dt < N^{-1}.
\]

Finally,

\[
\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} g(t) \, dt = \infty = g(x).
\]

\[
\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt = 0 = f(x).
\]

This completes the proof.

When \(m(B) = 1\), we do not know if our argument can be modified to make \(f = 0\) on \([0,1]\) except in [2]. Perhaps this requires an approach altogether different from ours.

We say that \(x\) is a knot point of the function \(F\) if its Dini derivatives satisfy

\[
D^+ F(x) = D^- F(x) = \infty \quad \text{and} \quad D_+ F(x) = D_- F(x) = -\infty.
\]

We conclude by presenting a simple and direct example of functions \(g_1\) and \(g_2\) having derivatives (finite or infinite) at every point such that \(g_1 - g_2\) has knot points in every interval. (Consult [4] for analogous examples.)

Let \(\{a_i\}_{i=1}^{\infty}\) and \(\{b_i\}_{i=1}^{\infty}\) be countable dense subsets of \((0,1)\) that are disjoint. Let \(Z(c,d,x) = \int_{0}^{x} \theta(c(t-d)) \, dt\) for \(c > 0\), \(d > 0\), \(x > 0\). We integrate to obtain

\[
Z(c,d,x) = \left\{ \begin{array}{ll}
2c^{-1}[1+(cd)^1] - 1 & \text{if } x < d, \\
2c^{-1}[1+(cd)^1] + (1+cd-cd)^{-1} - 2 & \text{if } x > d.
\end{array} \right.
\]

Let \(u_i\) denote the distance from \(a_n(i)\) to the set \(\{b_1,\ldots,b_i\}\), and let \(v_i\) denote the distance from \(b_n(i)\) to the set \(\{a_1,\ldots,a_i\}\). Put

\[
\begin{align*}
g_1(x) &= \sum_{i=1}^{\infty} Z(2^{-1}u_i^{-1},a_n(i),x), \\
g_2(x) &= \sum_{i=1}^{\infty} Z(2^{-1}v_i^{-1},b_n(i),x).
\end{align*}
\]

for \(0 < x < 1\). By the argument in the proof of Theorem 1 we prove that \(g_1\) and \(g_2\) are absolutely continuous functions on \((0,1)\) with \(g_1' = \infty\) on \(A\), \(g_2' = \infty\) on \(B\), and \(g_2'\) finite on \(A\). Put \(g = g_1 - g_2\). Then \(g\) is absolutely continuous on \((0,1)\), \(g' = \infty\) on \(A\) and \(g' = -\infty\) on \(B\). Each of the sets

\[
E_1 = \{x: D^+ g(x) = \infty\}, \quad E_2 = \{x: D^- g(x) = \infty\}, \quad E_3 = \{x: D_+ g(x) = -\infty\}, \quad E_4 = \{x: D_- g(x) = -\infty\}
\]

is a dense \(G_\delta\)-subset of \((0,1)\), i.e., is the intersection of countably many open dense subsets of \((0,1)\). It follows that \(E_1 \cap E_2 \cap E_3 \cap E_4\) is also a dense \(G_\delta\)-subset of \((0,1)\). But each point in this intersection is a knot point of \(g\), even though \(g_1\) and \(g_2\) have derivatives (finite or infinite) everywhere by the proof of Theorem 1.

REFERENCES


