DECAY OF SOLUTIONS OF A SYSTEM OF NONLINEAR KLEIN-GORDON EQUATIONS

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ABSTRACT. We study the asymptotic behavior in time of the solutions of a system of nonlinear Klein-Gordon equations. We have two basic results: First, in the $L^2(\mathbb{R}^3)$ norm, solutions decay like $O(t^{-3/2})$ as $t \to +\infty$ provided the initial data are sufficiently small. Finally we prove that finite energy solutions of such a system decay in local energy norm as $t \to +\infty$.

KEY WORDS AND PHRASES. Nonlinear Klein-Gordon equations, decay, local energy, uniform decay.

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1. INTRODUCTION

Our main purpose in this work will be to study time decay properties of solutions of the nonlinear system of Klein-Gordon equations

\[
\begin{align*}
\Box u + m^2 u + g^2 uv^2 &= 0 \\
v + \sigma^2 v + g^2 vu^2 &= 0
\end{align*}
\]

where $x$ runs in $\mathbb{R}^3$ and $t \geq 0$. Here $\Box$ denotes the d'Alembertian operator i.e. $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ and $\Delta$ is the usual Laplacian operator. In (1.1)-(1.2), $m$, $\sigma$ and $g$ are positive constants. Such systems of interacting relativistic (scalar) fields were suggested by a number of authors in the last two decades, among them we can mention I. Segal [1], K. Jörgens [2] and more recently, V.G. Makhankov [3].
In section 3 we consider solutions of (1.1)-(1.2) in a suitable Banach space $X$ and we prove, in particular, that $\|u(-,t)\|_{L^\infty(R^3)} = O(t^{-3/2})$ and $\|v(-,t)\|_{L^\infty(R^3)} = O(t^{-3/2})$ as $t \to \infty$, provided the initial data is small enough in an appropriate sense. This seems to be the best possible rate of decay (in the norm $\| \cdot \|_{L^\infty}$) for finite energy solutions of system (1.1)-(1.2). In order to obtain our result we use techniques which are essentially in the framework of contraction type notions together with known facts of the linear Klein-Gordon equation in three dimensional space.

In section 4 we study the local energy behavior as $t \to \infty$ for finite energy solutions of (1.1)-(1.2). The important work of C. Morawetz [4] was the starting point for our analysis in this section. Appropriate adaptations of [4] as well as the work of W.A. Strauss [5] to our case were needed. Unfortunately, we could not find the precise rate of decay in this case, which we suspect should be $O(t^{-1})$.

2. NOTATION AND PRELIMINARIES

In what follows we shall use standard notation: By $L^p(R^3)$, $1 \leq p < \infty$ we denote the space of functions in $R^3$ whose $p$th powers are integrable, with the norm $\|f\|_{L^p} = \left( \int |f(x)|^p dx \right)^{1/p}$ and by $L^\infty(R^3)$ we denote the space of measurable essentially bounded functions in $R^3$, with the norm $\|f\|_{L^\infty} = \text{ess sup} |f(x)|$. From now on, an integral sign to which no domain is attached will be understood to be taken over all space $R^3$. We denote by $\text{grad} u$ the gradient of $u$ (in space variables) and $\text{grad} u = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}$. The radial derivative (with respect to the origin) will be denoted by $u_r = \frac{x}{r} \cdot \text{grad} u$ where $r = |x|$. The Laplacian operator is denoted by $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$. For any positive integer $k$ and $1 \leq s \leq \infty$ we consider the Sobolev space $W^k, s(R^3)$ of (classes of) functions in $L^s(R^3)$ which together with their partial derivatives up to order $k$ belong to $L^s(R^3)$. The norm in $W^k, s(R^3)$ will be denoted by $\| \cdot \|_{W^k, s}$.

In case $s=2$ we shall write $H^k(R^3)$ instead of $W^{k, 2}(R^3)$. From now on, in order to simplify the notation we will denote by $C$ various constants (which may vary line to line). All functions considered in this paper are real-valued.

Since the system (2.2)-(1.2) is reversible in time, we shall perform our estimates only for $t>0$ and the same conclusion will be true for $t<0$. Most of the lemmas, especially in section 4, are proved only for the case in which the initial data at $t=0$ belongs to $C^\infty_0(R^3)$ (that is, the space of $C^\infty$ functions defined in $R^3$ with compact support). By a standard approximation procedure the same conclusion will be true for finite energy solutions.

Let us recall briefly some known facts concerning the linear problem: Consider the Klein-Gordon equation

$$\square \omega + m^2 \omega = 0 \quad , \quad x \in R^3 \quad , \quad t \in [0, \infty) \quad (2.1)$$

$$\omega(x,0) = \phi_1(x) \quad , \quad \omega_t(x,0) = \phi_2(x)$$

where $\square = \frac{\partial^2}{\partial t^2} - \Delta$ and $m>0$. Then, we have the following estimate (see [6]):
provided the initial data \((\phi_1, \phi_2)\) belong, say to \(C^\infty_0(\mathbb{R}^3)\).

Now, consider the inhomogeneous Klein-Gordon equation

\[
u + m^2 u = F(x,t), \quad x \in \mathbb{R}^3, \quad t \in [0, \infty)
\]

\(u(x,0) = 0 = u_t(x,0)
\)

where \(F \in L^1(0,T; H^p(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3))\), then using Duhamel's principle and (2.2) we obtain

\[
\left\| u(\cdot,t) \right\|_{L^\infty} \leq C \int_0^t (1+|t-s|)^{-\frac{3}{2}} \left\| F(\cdot,s) \right\|_{W^{1,1}} ds
\]

and

\[
\left\| u(\cdot,t) \right\|_{H^2} \leq C \int_0^t \left\| F(\cdot,s) \right\|_{H^p} ds
\]

Let us define the space of functions which will be using in the next section: Let \(\omega(x,t)\) be such that, for each \(t\) we have that \(\omega(\cdot,t) \in H^p(\mathbb{R}^3)\). We consider the norm \(\| \cdot \|_D\) defined by

\[
\left\| \omega \right\|_D^2 = \sup_{t \geq 0} \left( \left\| \omega(\cdot,t) \right\|_{H^p(\mathbb{R}^3)}^2 + (1+t)^3 \left\| \omega(\cdot,t) \right\|_{W^{1,\infty}}^2 \right)
\]

Let

\[X = \{ (u,v) \text{ such that } u(\cdot,t), v(\cdot,t) \in H^p(\mathbb{R}^3) \text{ and } \| u \|_D < \infty, \| v \|_D < \infty\}\]

In \(X\) we consider the norm \(\|(u,v)\|_D^2 = \| u \|_D^2 + \| v \|_D^2\). Clearly, \(X\) is a Banach space with the norm \(\| (\cdot, \cdot) \|\). Now, we indicate some simple lemmas which will be used in the next section

**Lemma 1.** Let \((u_1,v_1), (u_2,v_2) \in X\), then

\[
\left\| u_1v_1^2 - u_2v_2^2 \right\|_{H^p} + \left\| u_1v_1^2 - u_2v_2^2 \right\|_{W^{1,1}} \leq C \left\| u_1 - u_2 \right\|_D \left( \left\| v_1 \right\|_{W^{1,\infty}}^2 + \left\| v_1 \right\|_{W^{1,\infty}} \left\| v_1 \right\|_{H^p} \right) +
\]

\[
+ C \left\| v_1 - v_2 \right\|_D \left( \left\| u_2 \right\|_{W^{1,\infty}} + \left\| v_1 \right\|_{W^{1,\infty}} \left\| v_2 \right\|_{W^{1,\infty}} \right)
\]

**Proof.** Since \(H^p(\mathbb{R}^3)\) is an algebra, then, for each \(t\), \(u_1v_1^2 - u_2v_2^2 \in H^p(\mathbb{R}^3)\). The triangle inequality implies

\[
\left\| u_1v_1^2 - u_2v_2^2 \right\|_{H^p} \leq \left( \left\| u_1 - u_2 \right\|_{H^p} \left\| v_1^2 \right\|_{H^p} + \left\| u_2 \right\|_{H^p} \left\| v_1^2 - v_2^2 \right\|_{H^p} \right)
\]

Using the Leibnitz's rule and the imbedding \(H^p(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)\) we obtain
\[ \| (u_1 - u_2) v^2 \|_{H^3} \leq C \| u_1 - u_2 \|_D \left( \| v_1 \|_{W^{1,\infty}}^{\tau} \| v_1 \|_{W^{1,\infty}} \| v_1 \|_{H^3} \right) \]

and

\[ \| u_2 (v_1^2 - v_2^2) \|_{H^3} \leq C \| v_1 - v_2 \|_D \left( \| u_1 \|_{W^{1,\infty}} \| v_1 \|_{B^*} \| v_1 \|_{B^*}^{\tau} \| u_2 \|_{H^3} \| v_1 \|_{W^{1,\infty}} \right) \]

which together with (2.6) implies that we got the desired bound for the first term on the left hand side of the lemma. The estimate for the term \( \| u_1 v_1^2 - u_2 v_2^2 \|_{W^{3,1}} \) can be done similarly.

**Lemma 2.** Let \( q \geq 1, \, \, r > 0 \) such that \( r q > 1 \), then, for any \( t > 0 \) we have

\[ \int_0^\infty (1+|t-s|)^{-r} (1+s)^{-q} \, ds \leq C (1+t)^{-r} \]

**Proof.** See [7].

### 3. Decay as \( t \to \infty \) for Small Data

In this section we present a result concerning the asymptotic behavior for solutions of (1.1)-(1.2) in the space \( X \) and small initial data.

**Lemma 3.** Let \( (u_1, v_1), (u_2, v_2) \in X \) and \( \rho > 0 \). Suppose that \( \| (u_1, v_1) \|_{\rho} \leq \rho, \| (u_2, v_2) \|_{\rho} \leq \rho \).

Define the nonlinear maps \( N_m \) and \( N_0 \) by

\[ N_m [u, v] (x, t) = -g_2 \int_0^t R_m (x - y, t - s) u^j v^j dy ds \]

and

\[ N_0 [v, u] (x, t) = -g_2 \int_0^t R_0 (x - y, t - s) v^j u^j dy ds \]

\( j = 1, 2 \), where \( R_m \) and \( R_0 \) denote the Riemann functions associated with the linear Klein-Gordon operator \( \square + m^2 I \) and \( \square + \alpha^2 I \) respectively.

a) \( \| N_m [u_1, v_1] - N_m [u_2, v_2] \|_{B} \leq C \rho \| (u_1 - u_2, v_1 - v_2) \|_{\rho} \)

and

\[ \| N_0 [v, u] - N_0 [v, u] \|_{B} \leq C \rho \| (u_1 - u_2, v_1 - v_2) \|_{\rho} \]

**Proof.** Since \( H^3 (\mathbb{R}^3) \) is an algebra then it follows that for each \( t, u_1 v_1^2 - u_2 v_2^2 \in \mathcal{H}^3 (\mathbb{R}^3) \).

Using the definition and (2.4) we obtain, for each \( t \):

\[ \| N_m [u_1, v_1] - N_m [u_2, v_2] \|_{H^3} \leq C \int_0^t \| u_1 v_1^2 - u_2 v_2^2 \|_{H^3} \, ds \]

By lemma 1 it follows that

\[ \int_0^t \| u_1 v_1^2 - u_2 v_2^2 \|_{H^3} \, ds \leq C \| u_1 - u_2 \|_D [\| v_1 \|_{W^{1,\infty}}^{\tau} + \| v_1 \|_{W^{1,\infty}}^{\tau} \| v_1 \|_{H^3}] ds + \]

\[ + C \| v_1 - v_2 \|_D [\| v_1 \|_{B^*} + \| v_2 \|_{B^*}] \int_0^t \| u_2 \|_{W^{1,\infty}} ds + C \| v_1 - v_2 \|_D \| u_1 \|_D \int_0^t \| v_1 \|_{W^{1,\infty}} + \]

\[ + \| v_2 \|_{W^{1,\infty}}] ds \leq C \| u_1 - u_2 \|_D [\| v_1 \|_{B^*} + C \| v_1 - v_2 \|_D \| v_1 \|_D + \| v_2 \|_D] \| u_2 \|_D \]
Thus

\[ \| N_m[u_1,v_1] - N_m[u_2,v_2] \|_{H^k} \leq C \rho^2 [ \| u_1-u_2 \|_B + \| v_1-v_2 \|_B ] \]  

(3.3)

Using the definition and (essentially) (2.3) it follows that

\[ \| N_m[u_1,v_1] - N_m[u_2,v_2] \|_{W^{1,\infty}} \leq C \int_0^t (1+|t-s|)^{-3/2} \| u_1v_2-u_2v_1 \|_{W^{3,1}} ds \]

By lemmas 1 and 2 we deduce that

\[ \int_0^t (1+|t-s|)^{-3/2} \| u_1v_2-u_2v_1 \|_{W^{3,1}} ds \leq \{ C \| u_1-u_2 \|_B + \| v_1-v_2 \|_B \} \int_0^t (1+|t-s|)^{-3/2} (1+s)^{-3/2} ds \]

\[ \leq C (1+t)^{-3/2} \{ \| u_1-u_2 \|_B + \| v_1-v_2 \|_B \} \]

Consequently

\[ (1+t)^{3/2} \| N_m[u_1,v_1] - N_m[u_2,v_2] \|_{W^{1,\infty}} \leq C \rho^2 [ \| u_1-u_2 \|_B + \| v_1-v_2 \|_B ] \]  

(3.4)

Combining (3.3) with (3.4) we conclude item a). The proof of item b) is done exactly in the same fashion.

**Lemma 4.** Let \( U_0(X,t), V_0(X,t) \) be solutions of the free Klein-Gordon equations

\[ \square u + m^2 u = 0 \quad \square v + \sigma^2 v = 0 \]

respectively with initial data at time \( t=0 \) so that \( (U_0, V_0) \in X \). Let us consider the sequence \( \{(U_n, V_n)\}_{n=0}^{\infty} \) defined by \( (U(0), V(0))=(U_0, V_0) \) and

\[ U(n+1) = U_0 + N_m[U(n), V(n)] \]

\[ V(n+1) = V_0 + N_0[V(n), U(n)] \]

for \( n=1,2,... \) where \( N_m \) and \( N_0 \) were defined in (3.1). Then \( (U(n+1), V(n+1)) \in X \) for all \( n=0,1,2,... \).

**Proof.** The proof is done by induction. It is enough to prove that

\( (N_m[U(n), V(n)], N_0[V(n), U(n)]) \in X \) provided that \( (U(n), V(n)) \in X \). But this was already done during the proof of lemma 3. Consequently the conclusion of the lemma holds.

Now let \( U_0 \) and \( V_0 \) as in lemma 4 with initial data

\[ U_0(x,0) = \phi_1(x), \quad \frac{\partial U_0}{\partial t}(x,0) = \phi_2(x), \]

\[ V_0(x,0) = \psi_1(x) \quad \text{and} \quad \frac{\partial V_0}{\partial t}(x,0) = \psi_2(x) \]

such that \( \phi_j, \psi_j \in C^0(\mathbb{R}^3), j=1,2 \). Using (2.2) we can estimate the norm \( \|(U_0, V_0)\|_p \), say \( \|(U_0, V_0)\|_p \leq \rho \), where
On the other hand, let us choose \( \tilde{\rho} > 0 \) small enough so that \( \tilde{\rho} \leq \frac{1}{2\sqrt{2}} C \) where \( C > 0 \) is the constant which appears in the right hand side of inequality (3.2).

**Theorem (Decay for small data).** Let \( \phi_j, \psi_j \in C_0(\mathbb{R}^3) \), \( j = 1, 2 \) be chosen so that

\[
0 < \rho_0 = C[\phi_1, \psi_1]_{H^2} + [\phi_2, \psi_2]_{H^2} + [\phi_3, \psi_3]_{H^3} + [\phi_4, \psi_4]_{L^4, 1} + [\phi_5, \psi_5]_{L^4, 1}
\]

Then the sequence of successive approximations \( \{(u_n, v_n)\}_{n=0}^\infty \) defined in Lemma 3 converges to a pair \( (u, v) \in X \), which is a solution of (1.1)-(1.2) such that

\[
\begin{align*}
 u(x, 0) &= \phi_1(x), \quad u_t(x, 0) = \phi_2(x), \\
 v(x, 0) &= \psi_1(x), \quad v_t(x, 0) = \psi_2(x).
\end{align*}
\]

In particular, we have that \( \|u^{(n)}, v^{(n)}\|_{L^\infty_0(1+t)^{-3/2}} \leq \tilde{\rho} \) for all \( n = 0, 1, 2, \ldots \). If \( n = 0 \) this is trivial. Suppose that \( \|u^{(n)}, v^{(n)}\|_{L^\infty_0(1+t)^{-3/2}} \leq \tilde{\rho} \). Using the definition of \( u^{(n+1)} \) and \( v^{(n+1)} \) we obtain

\[
\begin{align*}
\|u^{(n+1)}(t), v^{(n+1)}(t)\|_{H^0}^2 + \|v^{(n+1)}(t), v^{(n+1)}(t)\|_{H^0}^2 &\leq 2(\|u^{(n)}(t), v^{(n)}(t)\|_{H^0}^2 + \|v^{(n)}(t), v^{(n)}(t)\|_{H^0}^2) + \\
&+ 2([N_m(u^{(n)}, v^{(n)})]_{H^0} + [N_o(u^{(n)}, v^{(n)})]_{H^0}) (\| u^{(n)}(t), v^{(n)}(t)\|_{H^0}^2)
\end{align*}
\]

and

\[
\begin{align*}
(1+t)^3 \| (u^{(n+1)}(t), v^{(n+1)}(t))_{H^1_x}^2 + \| (v^{(n+1)}(t), v^{(n+1)}(t))_{H^1_x}^2 &\leq 2(1+t)^3 \| (u^{(n)}(t), v^{(n)}(t))_{H^1_x}^2 + \| (v^{(n)}(t), v^{(n)}(t))_{H^1_x}^2) + \\
&+ 2([N_m(u^{(n)}, v^{(n)})]_{H^3, 1} + [N_o(u^{(n)}, v^{(n)})]_{H^3, 1}) (\| u^{(n)}(t), v^{(n)}(t)\|_{H^0}^2)
\end{align*}
\]

From (3.5) and (3.6) we conclude that

\[
\begin{align*}
\| (u^{(n+1)}, v^{(n+1)}) \|_{L^\infty_0(1+t)^{-3/2}}^2 &\leq 2 \| (u^{(n)}, v^{(n)}) \|_{L^\infty_0(1+t)^{-3/2}}^2 + 2(C\tilde{\rho})^2 \| (u^{(n)}, v^{(n)}) \|_{L^\infty_0(1+t)^{-3/2}}^2 \\
&\leq 2(\frac{\tilde{\rho}}{2})^2 + 2(\frac{1}{2\sqrt{2}})^2 \tilde{\rho}^2 \leq \tilde{\rho}^2
\end{align*}
\]

because our choice of \( \tilde{\rho} \). This concludes the proof of our claim. For any positive integer \( n \) we define

\[
e_n = \| (u^{(n+1)} - u^{(n)}, v^{(n+1)} - v^{(n)}) \|
\]

Consequently we have \( e_n \leq C_\rho^2 e_{n-1} \) because of lemma 3 and the above observation. By iteration it follows that \( e_n \leq \left( \frac{\sqrt{2}}{C_\rho^2} \right)^n e_0 \leq 2^{-n} e_0 \). Now, let \( k > n \). Using the above observation we conclude that
as \( k, n \to \infty \).

Thus, there exist a pair \((u, v) \in X\) such that \((u(n), v(n)) \to (u, v)\) in \( X \) as \( n \to \infty \). Obviously \( \| (u, v) \| \leq \delta \). Thus, by lemma 3 it follows that \( \| N_m(u(n), v(n)) - N_m(u, v) \| D \to 0 \) as \( n \to \infty \) and \( \| N_\sigma^m[v(n), u(n)] - N_\sigma^m[v, u] \| D \to 0 \) as \( n \to \infty \). Consequently \((u, v) = (u_0, v_0) + (N_m[u, v], N_\sigma[v, u])\) so that the pair \((u, v)\) is a solution of system (1.1)-(1.2).

4. LOCAL ENERGY DECAY

In this section we consider finite energy solutions of the system (1.1)-(1.2) without our previous assumptions of smallness on the initial data.

We shall concentrate our attention on the local energy \( E_\Omega(t) \) associated with the pair \((u, v)\):

\[
E_\Omega(t) = \frac{1}{2} \int_\Omega \left[ u_t^2 + \sum \text{grad } u_t^2 + \sum \text{grad } v_t^2 + g^2 u^2 + g^2 v^2 \right] dx
\]

where \( \Omega \) is a bounded region of \( \mathbb{R}^3 \). In many practical situations \( \Omega \) can be assumed be a ball. We will show in this section that under suitable assumptions on the initial data of the system (1.1)-(1.2) then \( E_\Omega(t) \) approaches zero as \( t \to +\infty \).

Our analysis is based on the work of C. Morawetz [4] where she studied a single nonlinear equation. First, we present an existence result: Let us consider the space \( Y=H^1(\Omega) \times H^1(\Omega) \) and the matrix differential operator \( A \) given by

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\Delta - m^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \Delta - \sigma^2 & 0
\end{pmatrix}
\]

We can rewrite (1.1)-(1.2) with \( u_1=u_t, u_2=u, v_1=v_t, v_2=v_t \) as a system of four equations of first order in time

\[
\frac{d\phi}{dt} = A\phi + N(\phi)
\]

where \( \phi=(u_1, u_2, v_1, v_2) ^T \), \( N(\phi)=(0, -g^2 u v_t, 0, -g^2 v u_t)^T \) (here \( (\cdot)^T \) means the transpose of \( (\cdot) \)). Clearly \( A \) is skew-adjoint with domain \( D(A)=H^3(\Omega) \times H^3(\Omega) \times \mathbb{R}^3 \times \mathbb{R}^3 \).

**Lemma 5.** For any \( \phi, \psi \in Y \) we have

\[
\| N(\phi) - N(\psi) \|_Y \leq C(\| \phi \|_Y, \| \psi \|_Y) \| \phi - \psi \|_Y
\]

where \( C \) is an increasing function of norms \( \| \phi \|_Y \) and \( \| \psi \|_Y \).

**Proof.** Let \( \phi=(u_1, u_2, v_1, v_2)^T \) and \( \psi=(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)^T \in Y \). Since \( H^j(\Omega) \) is an algebra then \( N(\phi), N(\psi) \in Y \). The triangle inequality implies that

\[
\| N(\phi) - N(\psi) \|_Y \leq g^2 \| u_1 v_2 - \tilde{u}_1 \tilde{v}_2 \|_L^2 + g^2 \| v_1 u_2 - \tilde{v}_1 \tilde{u}_2 \|_L^2 + 2g^2 \| (u_1 - \tilde{u}_1) v_t \|_L^2 + 2g^2 \| (v_1 - \tilde{v}_1) u_t \|_L^2.
\]
Holder's inequality followed by Sobolev's inequality give us
\[ \| N(\phi) - N(\psi) \|_q \leq C \left( \| u_1 - \tilde{u}_1 \|^2_{H^1} + \| v_1 \|^2_{H^1} \right) \]
\[ + C \left( \| v_1 - \tilde{v}_1 \|^2_{H^1} + \| \tilde{v}_1 \|^2_{H^1} \right) \leq C \left( \| \phi \|^2_{H^1} + \| \psi \|^2_{H^1} \right) \]

**LEMMA 6.** a) \( N: D(A) \to D(A) \) and b) \( \| A(N(\phi) - N(\psi)) \| \leq C \| \phi \|^2_{H^1} + \| \psi \|^2_{H^1} + \| A\phi \|^2_{H^1} + \| A\psi \|^2_{H^1} + \| A\phi - A\psi \|^2_{H^1} \)
for all \( \phi, \psi \in D(A) \).

**PROOF.** Item a) is trivial. Let \( \phi = (u_1, u_2, \tilde{u}_1, \tilde{v}_1) \) and \( \psi = (\tilde{u}_1, \tilde{v}_1, \tilde{v}_2, \tilde{v}_2) \) belonging to \( D(A) \). A direct calculation gives us
\[ \| (u_1^2 - \tilde{u}_1^2_v) \|^2_{H^1} \leq 2g^2 \| (u_1 - \tilde{u}_1) \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} \]
\[ \| (u_1^2 - \tilde{u}_1^2) \|^2_{H^1} \leq 2g^2 \| (u_1 - \tilde{u}_1) \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} + 2g^2 \| \tilde{v}_1 - \tilde{v}_2 \|^2_{H^1} \]

We use Hölder's and Sobolev's inequality to obtain
\[ \| (u_1 - \tilde{u}_1) \|^2_{H^1} \leq C \| A\phi - A\psi \|^2_{H^1} \]
\[ \| \tilde{u}_1 (\tilde{v}_1 - \tilde{v}_2) \|^2_{H^1} \leq C \| A\phi - A\psi \|^2_{H^1} \]
\[ \| (\tilde{v}_1 - \tilde{v}_2) \|^2_{H^1} \leq C \| A\phi - A\psi \|^2_{H^1} \]

Combining the last four inequalities with (4.3) we conclude the proof of the lemma.

**THEOREM 2 (Global existence).** Let the initial data at time \( t=0 \) for the system (1.1)-(1.2) belong to the subspace \( D(A) = H^2 \mathbb{R}^3 \). Then, there exist a (strong) solution of (4.2) for all time \( t \geq 0 \).

**PROOF.** According to Segal's theorem [6], lemmas 4 and 5 imply that there exists a unique local solution of (4.2) defined in a maximal interval \( I = (0, T_{\text{max}} \leq +\infty) \) of existence. Now, let us write (1.1)-(1.2) as

\[ \square \ u + m^2 u = f \]  
\[ \square \ v + \sigma^2 v = h \]

where \( f = -g^2 uv^2 \) and \( h = -g^2 vu^2 \). We can use the linear theory: Multiply (4.4) by \( u_t \) and (4.5) by \( v_t \). Next, integration in the whole space give us

\[ \frac{1}{2} \frac{d}{dt} \left[ u_t^2 + |\text{grad } u|^2 + m^2 u^2 \right] = \int f u_t \ dx \]  
\[ \frac{1}{2} \frac{d}{dt} \left[ v_t^2 + |\text{grad } v|^2 + \sigma^2 v^2 \right] = \int h v_t \ dx \]

But \( \int (f u_t + h v_t) \ dx = - \frac{g^2}{2} \frac{d}{dt} \int u_t v_t \ dx \). Adding the identities (4.6) we conclude that

\[ E_\infty(t) = \frac{1}{2} \left[ |u_t|^2 + |\text{grad } u|^2 + m^2 u^2 + v_t^2 + |\text{grad } v|^2 + \sigma^2 v^2 + g^2 u_t v_t \right] \ dx = \text{Constant} \]
in the interval $I$. In particular, this implies that $\| \phi(t) \|_Y$ is bounded for all $t \in I$. This concludes the proof of the theorem.

**REMARKS.** Using essentially the same procedure as above one can prove higher regularity of the solutions provided that the initial data is more regular. If the initial data belongs to $H^{j+1}_0 \Theta H^{j+1}_0 \Theta H^{j+1}_0 (\mathbb{R}^3)$ then the solution pair of (1.1)-(1.2) will belong to $[C(I; H^{j+1}_0 (\mathbb{R}^3))]^2$.

**LEMMA 7.** Let $(u,v)$ be the solution of the system (1.1)-(1.2) with initial data belonging to $[C^0_0(\mathbb{R}^3))^2$. Then, for any $T > 0$ and $y \in \mathbb{R}^3$ we have

$$\int_0^T [u^2(y,t)+v^2(y,t)] dt \leq C E_\infty(0)$$

where $C > 0$ is independent of $T$ and $E_\infty(0) = E\_\infty(0)$ is given by (4.1) with $\Omega = \mathbb{R}^3$.

**PROOF.** Let $y \in \mathbb{R}^3$. For any $x \neq y$ let us denote by $r = |x-y|$ and $\frac{\partial}{\partial r} = \frac{x-y}{r}$ grad. We consider Morawetz's multiplier $M(u) = \frac{\partial u}{\partial r} + \frac{u}{r}$. Multiply (1.1) by $M(u)$ and (1.2) by $M(v)$. Adding those two expressions we obtain after some calculations

$$0 = ( \square u + m^2 u + g^2 uv^2) M(u) + ( \square v + \epsilon^2 v + g^2 uu^2) M(v) = \frac{\partial A}{\partial r} + \text{div} B + D$$

where $A = u \cdot M(u) + v \cdot M(v)$ and

$$B = [m^2 u + \epsilon^2 v + g^2 uu^2 + g^2 uv^2] \text{grad} u^2 + [\text{grad} v^2 - u^2_t - \frac{u}{r^2} - \frac{v}{r^2}] \frac{(x-y)}{2r} -$$

$$- [M(u) \text{grad} u + M(v) \text{grad} v]$$

and

$$D = \frac{1}{r} [ \text{grad} u^2 - u^2 + \text{grad} v^2 - v^2 + g^2 uu^2]$$

Integration in $\mathbb{R}^3$ of the identity (4.8) give us

$$\frac{d}{dt} \int A(x,t) dx - \int \text{div} [(x-y) \frac{u^2 + v^2}{2r^3}] dx + D(x,t) dx = 0$$

Since $D \geq 0$ we obtain

$$\frac{d}{dt} \int A(x,t) dx + 2\pi u^2(y,t) + 2\pi v^2(y,t) \leq 0$$

Integration from $t=0$ to $t=T>0$ gives us

$$\int_0^T [u^2(y,t)+v^2(y,t)] dt \leq \frac{1}{2\pi} \int [A(x,0) - A(x,T)] dx$$

(4.9)

Let us estimate $\int A(x,t) dx$. The following simple inequalities are useful:

$$\pm 2M(u) u_t \leq u_r^2 + \frac{u^2}{r} + \frac{2u^2}{r^2} + u_t^2 = \text{div} (u^2) + u_r^2 + u_t^2$$

$$\pm 2M(v) v_t \leq \text{div} (v^2) + v_t^2 + v_r^2$$
Thus, for any $t \geq 0$ we have

$$
\int A(x,t) dx \leq \int (u_r^2 + u_t^2 + v_r^2 + v_t^2) dx \leq 2E_\infty(0)
$$

(4.10)

From (4.9) and (4.10) we obtain the conclusion of the lemma.

**LEMMA 8.** Let $\omega: \mathbb{R}^3 \to \mathbb{R}$ be a $C^1$ function and $y \in \mathbb{R}^3$. Then

a) $\omega \leq |\nabla \omega(x)|^2 - \omega_r^2(x)$ for any $x$ such that $|x-y|=r$. Here $\nabla \omega(x) = \nabla \omega(x)$ and $\omega_r(x) = \frac{(x-y)}{r} \nabla \omega(x)$ where $\nabla \omega(x)$ denotes a (unit) tangent vector at $x$. \hfill (4.11)

b) $|\nabla \omega(x)| \leq \frac{3}{2} \omega_r^2(x)$ for any $x \in \mathbb{R}^3$ where $\tau_1$, $\tau_2$, and $\tau_3$ are (unit) tangent vectors to the spheres $S_j = \{ \xi \in \mathbb{R}^3 \text{ such that } |\xi - \xi_j| = |x - \xi_j| \}$ for $j = 1, 2, 3$ respectively, for some convenient choice of $\xi_1$, $\xi_2$, and $\xi_3$. \hfill (4.12)

**PROOF.** Given $\tau(x)$ let us choose another vector $\tau_0$ so that $\{\tau(x), \tau_0, \eta\}$ are orthonormal. Here $\eta$ denotes a vector in the direction of the radius $r = |x-y|$. Now, it is clear that $\omega_t^2 + \omega_r^2(x) \leq |\nabla \omega(x)|^2$. This proves item a). Let $\xi_j$, $j = 1, 2, 3$ and three planes $P_j$, $j = 1, 2, 3$ so that $x \in P_1 \cap P_2 \cap P_3$ and their normal vectors are $x - \xi_j$, $j = 1, 2, 3$ respectively. Let $\tau_j(x) \in P_j$ be (unit) tangent vectors to the spheres $S_j = \{ \xi \in \mathbb{R}^3 \text{ such that } |\xi - \xi_j| = |x - \xi_j| \}$ so that they are linearly independent and the angle between $\nabla \omega(x)$ and $\tau_j(x)$ is less or equal to $\pi/2$. Then we can write $\nabla \omega(x)$ as a linear combination of the $\tau_j(x)$'s, $j = 1, 2, 3$ with nonnegative coefficients.

Therefore, $|\nabla \omega(x)| \leq \frac{3}{2} \omega_r^2(x)$ which implies $|\nabla \omega(x)| \leq \frac{3}{2} \omega_r^2(x)$. \hfill (4.13)

**LEMMA 9.** Let $(u,v)$ be the solution of system (1.1)-(1.2) with initial data at time $t=0$ belonging to $[C_0^\infty(\mathbb{R}^3)]^n$. Let $m, \Omega \subset \mathbb{R}^3$ a bounded region, then for any $T > 0$ we have

a) $\int_0^T \int_\Omega \left[ |\nabla u|^2 + |\nabla v|^2 + g^2 u^2 v^2 \right] dx dt \leq C(\Omega)E_\infty(0)$

b) $\int_0^T E_\infty(t) dt \leq C(\Omega)E_\infty(0)$

where $C(\Omega)$ is a positive constant independent of $T$.

**PROOF.** a) We use identity (4.8). Integration in the whole space gives us, for any $y \in \Omega$:

$$
\int_\Omega D(x,t) dx \leq 2\pi [u^2(y,t) + v^2(y,t)] \int_\Omega D(x,t) dx = -\frac{d}{dt} \int_\Omega A(x,t) dx
$$

Therefore, integration in time from $t=0$ to $t=T$ implies that

$$
\int_0^T \int_\Omega D(x,t) dx dt \leq C(\Omega)E_\infty(0)
$$

(4.11)

because we have used our previous estimate (4.10).

Let $d = \text{diameter of } \Omega$ and $\rho > d \Rightarrow |x-y|$. Thus, from (4.11) we obtain

$$
\frac{1}{\rho} \int_0^T \int_\Omega r D(x,t) dx dt \leq C(\Omega)E_\infty(0)
$$
Therefore

\[
\int_0^T \left[ |\text{grad } u|^2 - u_t^2 + |\text{grad } v|^2 - v_t^2 + g^2 u^2 v^2 \right] dx dt \leq C \rho \varepsilon(0) \tag{4.12}
\]

Now, we use lemma 8 with \( y = \xi_j \), \( j = 1, 2, 3 \). By part a) and (4.12) we obtain

\[
\int_0^T \left\{ \frac{3}{2} \left( u_{t, j}^2 + v_{t, j}^2 \right) + g^2 u^2 v^2 \right\} dx dt \leq C \rho \varepsilon(0) \tag{4.13}
\]

Using part b) of lemma 8 and (4.13) we conclude the proof of part a).

It remains to obtain a bound for

\[
\int_0^T \{ m^2 u_t^2 + u_t^2 + \sigma^2 v_t^2 + v_t^2 \} dx dt
\]

Let \( \alpha > 0 \) and h: \((0, \infty) \to \mathbb{R}\) a C\(^\infty\) function such that 1) \( h(0) = \alpha \), 2) \( h \equiv 0 \) for all \( s \geq \alpha \) and 3) \( h'(s) < 0 \) for all \( 0 < s < \alpha \).

Let \( y \in \mathbb{R}^3 \) and x = y. Denote by \( r = |x - y| \). First, we multiply identity (4.8) by \( h(r) \) and then we integrate in space to obtain

\[
0 = 2\pi \alpha [u^2(y, t) + v^2(y, t)] - \int h'(r)B^*(x-y) \frac{(x-y)}{r} dx + \frac{d}{dt} \int h(r)A(x, t)dx + \int h(r)D(x, t)dx \tag{4.14}
\]

The following identity can be easily verify

\[
2B^* \left( \frac{x-y}{r} \right) = g^2 u^2 v^2 + |\text{grad } u|^2 + |\text{grad } v|^2 - 2u_t^2 - 2v_t^2 + m^2 u^2 + \sigma^2 v^2 - u_t^2 - u^2_t - v^2 - \frac{2uu}{r} - \frac{2vv}{r} r
\]

Substitution of (4.15) in (4.14) and then integration in time from \( t = 0 \) to \( t = T \) implies

\[
- \frac{1}{2} \int_0^T h'(r) \left[ u_t^2 + v_t^2 + \left( \frac{1}{r^2} - 1 \right) (m^2 u^2 + \sigma^2 v^2) + \frac{2uu}{r} + \frac{2vv}{r} r \right] dx dt = \int_0^T \left[ u^2(y, t) + v^2(y, t) \right] dt - \int_0^T \left[ h'(r) [g^2 u^2 v^2 + |\text{grad } u|^2 + \right.
\]

\[ + |\text{grad } v|^2 - 2u_t^2 - 2v_t^2 \right] dx dt + \int_0^T \left[ h(r) [A(x, T) - A(x, 0)] dx + \right.
\]

\[ \left. + \int_0^T h(r)D(x, t)dx \right] \tag{4.16}
\]

Now, using lemma 7, (4.10) and (4.11) we deduce from (4.16) the following estimate

\[
- \int_0^T \frac{h'(r)}{2} \left[ u_t^2 + \left( \frac{1}{r^2} - 1 \right) (m^2 u^2 + \sigma^2 v^2) + v_t^2 + \frac{2uu}{r} + \frac{2vv}{r} r \right] dx dt \leq C \rho \varepsilon(0) +
\]

\[
+ C \max_{0 \leq \xi \leq \alpha} |h'(\xi)| \int_0^T \left[ g^2 u^2 v^2 + |\text{grad } u|^2 + |\text{grad } v|^2 - u_t^2 - v_t^2 \right] dx dt \tag{4.17}
\]
Finally, we use (4.12) to obtain from (4.17)

\[
- \frac{1}{2} \int_0^T h'(r) \left[ \frac{(u^2 + v^2)}{r} + \frac{(v^2 + u^2)}{r} + \frac{u^2 + v^2}{r} + \frac{(m^2 - 1)}{r^2} \right] \leq C \max_{0 \leq \xi \leq \alpha} |h'(\xi)|C(\Omega)E_\infty(0)
\]

Let us choose \( \alpha = \min\left( \frac{\sqrt{m^2 - 1}}{\sqrt{2}}, \frac{\sqrt{\sigma^2 - 1}}{\sigma} \right) = \alpha_0 \). Thus, if \( r = |x - y| \leq \alpha \) it follows that

\[
- \frac{1}{2} \int_0^T \int_{|x - y| \leq \alpha} h'(r) \left[ \frac{u^2 + v^2}{r} + \frac{u^2}{r} + \frac{v^2}{r} \right] \leq C(\Omega)E_\infty(0)
\]  

(4.18)

In particular

\[
- \frac{1}{2} \int_0^T \int_{|x - y| \leq \alpha/2} \frac{h'(r)}{2} \left[ \frac{u^2 + v^2}{r} + \frac{u^2}{r} + \frac{v^2}{r} \right] \leq C(\Omega)E_\infty(0)
\]

or

\[
\int_0^T \int_{|x - y| \leq \alpha/2} \left[ \frac{u^2 + v^2}{r} + \frac{u^2}{r} + \frac{v^2}{r} \right] \leq C(\Omega)E_\infty(0)
\]  

(4.19)

where \( \beta = \inf \left(-h'(r)\right) > 0 \)

(4.20)

where \( \Omega = \{x, |x - y| \leq \alpha/2\} \). Since \( \Omega \) is a bounded region we can cover it by a finite number of such balls. This implies part b).

**Theorem 3** (Decay of local energy). Let \((u, v)\) be the solution of system (1.1)-(1.2) with initial data belonging to \([C_0^\infty(\Omega)]\) and \(m, \sigma \geq 1\). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded region, then

a) \( \lim_{t \to \infty} \int_\Omega u^2(x, t)dx = 0 \) and b) \( \lim_{t \to \infty} E(t) = 0 \)

**Proof.** Let \( T > 0 \). We know by lemma 7 that

\[
\int_0^T (u^2 + v^2)dxdt \leq C(\Omega)E_\infty(0)
\]

Letting \( T \to \infty \) we obtain

\[
\int_0^\infty (u^2 + v^2)dxdt < +\infty
\]  

(4.20)

Let \( G(t) = \int_\Omega (u^2 + v^2)dx \). We also have
\[
\left| \frac{d}{dt} G(t) \right| \leq 2 \int \Omega (u_x v_x + v_x v_x) \, dx \leq \int \Omega (u^2 + u^2 + v^2 + v^2) \, dx \leq CE_\infty(0)
\]
(4.21)

From (4.20) and (4.21) it follows that \( \lim_{t \to +\infty} G(t) = 0 \). Consequently, \( \lim_{t \to +\infty} \int \Omega u^2 \, dx = \lim_{t \to +\infty} \int \Omega v^2 \, dx = 0 \).

Let us look back to the inequality (4.19) and let us fix \( \alpha_1, \alpha_2 \) so that \( 0 < \alpha_1 < \alpha_2 < \alpha_0 \). We define \( F(t) \) as
\[
F(t) = \int_{\Omega_{\alpha_1}} \int_{\Omega_{\alpha_2}} E_{\alpha} (t) \, dx
\]
(4.22)

where \( \Omega_{\alpha} = \{ x \in \mathbb{R}^3 / |x-y| < \alpha/2 \} \). Integration in time of (4.22) implies
\[
\int_0^T F(t) \, dt = \int_{\alpha_2}^{\alpha_1} \int_0^T E_{\alpha} (t) \, dt \leq CE_\infty(0)(\alpha_2 - \alpha_1)
\]

Therefore \( \int_0^\infty F(t) \, dt < +\infty \). A simple calculation shows that
\[
\left| \frac{d}{dt} F(t) \right| \leq \frac{1}{2} \int_{\alpha_1}^{\alpha_2} \int_{\Omega_{\alpha}} (u^2 + u^2 + v^2 + v^2) \, dx \leq E_\infty(0)
\]

which together with the above observations implies that \( \lim_{t \to +\infty} E_{\Omega}(t) = 0 \).

REFERENCES


