CONTINUOUS DEPENDENCE OF BOUNDARY VALUES FOR SEMIINFINITE INTERVAL ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Certain elliptic equations arising in catalysis theory can be transformed into ordinary differential equations on the interval \((0, \infty)\). The solutions to these problems usually depend on parameters \(\rho \in \mathbb{R}^n\), say \(u(t, \rho)\). For certain types of nonlinearities, we show that the boundary value \(u(\infty, \rho)\) is continuous on compact sets of the variable \(\rho\). As a consequence, bifurcation results for the elliptic equation are obtained.

KEY WORDS AND PHRASES. Continuous dependence, catalysis theory, bifurcation

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I. INTRODUCTION.

Let \(\varepsilon_0\) be a positive real number. Let \(\ell(\varepsilon)\) be a continuous function with domain \([0, \varepsilon_0]\) and range contained in \([-\infty, 0]\). Let \(S = \{(\varepsilon, u) \in \mathbb{R}^2 : 0 < \varepsilon < \varepsilon_0, \ell(\varepsilon) < u\}\).

Let \(f \in C^2(S)\) have the following properties:

\begin{align*}
\lim_{u \to 0^+} f(\varepsilon, u) &= -\infty \text{ for each } \varepsilon, f(\varepsilon, u) = -\infty \text{ on } [0, \varepsilon_0] \times [\mathbb{R} - S] \quad (1.1) \\
 f_u(\varepsilon, u) &\geq 0 \text{ and } f_{uu}(\varepsilon, u) \leq 0 \text{ on } S \quad (1.2) \\
 \lim_{\varepsilon \to 0^+} f(\varepsilon, u) &= u \text{ for each } u \quad (1.3)
\end{align*}

As a consequence of (1.3), we also have

\begin{equation}
\lim_{u \to \infty} f(\varepsilon, u) = L(\varepsilon) \geq 0 \text{ for each } \varepsilon \geq 0 \quad (1.4)
\end{equation}

We consider the semiinfinite interval initial value problem

\begin{align*}
\ddot{u} + \lambda e^{-2\varepsilon}\exp[f(\varepsilon, u)] &= 0, \quad 0 < t < \infty, \quad \lambda > 0 \quad (1.5) \\
u(0) &= \alpha, \quad \dot{u}(0) = \beta \quad (1.6)
\end{align*}

where \(f(\varepsilon, u)\) has the properties described in (1.1) through (1.4).

Some problems in catalysis theory (in two spatial dimensions) are modeled by (1.5)-(1.6) with the boundary condition \(\dot{u}(\infty) = 0\). The classic example is the case \(f(\varepsilon, u) = u(1 + cu)^{-1}\). The limiting case, \(f(0, u) = u\), gives us the Gelfand problem which can be solved explicitly in terms of elementary functions.
We prove that for solutions, \( u(t, \lambda, \alpha, \beta, \varepsilon) \), to (1.5)-(1.6), the boundary value \( \hat{u}(\omega, \rho) \) is continuous as a function of \( \rho = (\lambda, \alpha, \beta, \varepsilon) \) on compact sets with the property that \( \lambda \geq \lambda_0 > 0 \). As a consequence, a bifurcation result for (1.5) with boundary data \( u(0) = 0, \hat{u}(\omega) = 0 \), is obtained.

The methods for proving continuous dependence are also applicable to other types of nonlinearities where the bifurcation results (using \( f(0, u) \)) are much different than in the above problem.

2. PRELIMINARY LEMMAS.

The following lemmas are needed to prove the continuous dependence results for (1.5)-(1.6) at the boundary at \( \infty \).

**Lemma 1.** Let \( D = \{ (\lambda, \alpha, \beta, \varepsilon) : \lambda > 0, \beta > 0, 0 \leq \varepsilon \leq \varepsilon_0 \} \). For each \( \rho \in D \),

\[
\lim_{t \to \infty} \hat{u}(t, \rho) \text{ exists.}
\]

**Proof.** For each \( \rho \in D \), define \( \omega(\rho) = \sup \{ t \in [0, \infty) : \ell(\varepsilon) < u(t, \rho) \} \). Since \( \hat{u}(t, \rho) \leq 0 \), \( \hat{u}(t, \rho) \) is decreasing. If \( \hat{u}(t, \rho) > 0 \) for all \( t \geq 0 \), then \( \hat{u}(t, \rho) \) is bounded below and decreasing. Thus, \( \lim_{t \to \infty} \hat{u}(t, \rho) \) exists.

However, if \( \hat{u}(T, \rho) = 0 \) for some finite \( T \in [0, \omega) \), then \( u(t, \rho) \) attains a maximum value at \( u(T, \rho) \). But \( f(\varepsilon, u) \) is increasing in \( u \), so it is true that \( f(\varepsilon, u(t, \rho)) \leq f(\varepsilon, u(T, \rho)) =: \ell \). Equation (1.5) implies that

\[
\hat{u}(t, \rho) = -\lambda e^{-2t} \exp \{ f(\varepsilon, u(t, \rho)) \} \geq -\lambda ke^{-2t}
\]

(2.1)

So \( \hat{u}(t, \rho) \) is bounded below and decreasing. Thus, \( \lim_{t \to \infty} u(t, \rho) \) exists.

Notice that if \( \omega(\rho) < \infty \), then \( u(\omega(\rho), \rho) = \ell(\varepsilon), (1.5) \) becomes \( \hat{u} = 0 \) for \( t \geq \omega \), and \( \hat{u}(\omega(\rho), \rho) = \hat{u}(\omega(\rho)) \). In all cases, define \( m(\rho) = \hat{u}(\omega(\rho)) \).

**Lemma 2.** \( L(\varepsilon) \) is upper semicontinuous on \([0, \varepsilon_0]\).

**Proof.** Let \( \eta > 0 \) and \( \varepsilon_1 \in [0, \varepsilon_0] \) be given. There exists a number \( u_1 > 0 \) such that \( f_u(\varepsilon_1, u_1) = L(\varepsilon_1) + \frac{\eta}{2} \) for \( u > u_1 \) since \( f_u(\varepsilon_1, u) = L(\varepsilon_1) \) as \( u \to \infty \). Also, there is a number \( \delta > 0 \) such that \( f_u(\varepsilon_1, u_1) - \frac{\eta}{2} \leq f_u(\varepsilon_1, u_1) \) for \( |\varepsilon - \varepsilon_1| < \delta \) since \( f_u(\varepsilon, u_1) + f_u(\varepsilon_1, u_1) \) as \( \varepsilon \to \varepsilon_1 \). Finally, \( L(\varepsilon) \leq f_u(\varepsilon, u_1) \) since \( f_{uu} \leq 0 \) and since \( f_u(\varepsilon, u) = L(\varepsilon) \) as \( u \to \infty \). Combining these facts gives us

\[
L(\varepsilon) - \frac{\eta}{2} \leq f_u(\varepsilon, u_1) - \frac{\eta}{2} \leq f_u(\varepsilon_1, u_1) \leq L(\varepsilon_1) + \frac{\eta}{2}
\]

(2.2)

or, \( L(\varepsilon) \leq L(\varepsilon_1) + \eta \) for all \( \varepsilon \) such that \( |\varepsilon - \varepsilon_1| < \delta \). Thus, \( \lim_{\varepsilon \to \varepsilon_1} L(\varepsilon) \leq L(\varepsilon_1) + \eta \).

But \( \eta \) can be chosen arbitrarily small, so \( \lim_{\varepsilon \to \varepsilon_1} L(\varepsilon) \leq L(\varepsilon_1) \); that is, \( L(\varepsilon) \) is upper semicontinuous at \( \varepsilon_1 \). Since \( \varepsilon_1 \) was also arbitrary, \( L(\varepsilon) \) is upper semicontinuous on the interval \([0, \varepsilon_0]\).

**Lemma 3.** The value \( m(\rho) = \hat{u}(\omega(\rho), \rho) \) is upper semicontinuous on compact sets of the variable \( \rho \).

**Proof.** Let \( C \) be a compact subset of \( D \) and let \( \rho_0 \in C \). From lemma 1, for a given \( \eta > 0 \), there exist numbers \( \delta > 0 \) and \( T > 0 \) such that \( \hat{u}(T, \rho_0) \leq m(\rho_0) + \frac{\eta}{2} \) and \( \hat{u}(T, \rho) - \frac{\eta}{2} \leq \hat{u}(T, \rho_0) \) for \( |\rho - \rho_0| < \delta \) since \( \hat{u}(T, \rho) \) is continuous in \( \rho \) by standard continuous dependence. Also, \( m(\rho) \leq \hat{u}(T, \rho) \) since \( \hat{u}(t, \rho) \leq 0 \). Thus,
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\[ m(\rho) - \frac{\bar{c} \eta}{2} \leq \hat{u}(T, \rho) - \frac{\bar{c} \eta}{2} \leq \hat{u}(T, \rho_0) \leq m(\rho_0) + \frac{\bar{c} \eta}{2} \]

or, \( m(\rho) \leq m(\rho_0) + \eta \) for \(|\rho - \rho_0| < \delta\). As in the proof of lemma 2, it follows that \( \lim \rho \rightarrow \rho_0 \ m(\rho) \leq m(\rho_0) \); that is, \( m(\rho) \) is upper semicontinuous on compact sets of the variable \( \rho \).

**Lemma 4.** If \( L(\rho) > 0 \), then \( m(\rho) < 2/L \) on the set \( D \).

**Proof.** Integrating equation (1.5) yields

\[ m(\rho) + \int_0^\infty \lambda e^{-2t} \exp[f(\rho, u(t, \rho))] dt = \beta \]

By our assumptions on \( f \), it is a fact that \( f(\rho, u) \geq L(\rho) \), so \( f(\rho, u) \geq f(\rho, 0) + L(\rho)u \)

for \( u \geq 0 \). Suppose that for some \( \rho \in D \), \( m(\rho) > 2/L \). The conditions that \( m \) is finite and \( \bar{c} \leq 0 \) imply that \( u(t) \geq m(t) \) for \( t \geq 0 \). So

\[ \int_0^\infty \lambda e^{-2t} \exp[f(\rho, u)] dt \geq \int_0^\infty \lambda e^{-2t} e^{f(0)} \frac{d}{dt} \left[ u(t) \right] dt \geq \int_0^\infty \lambda e^{-2t} e^{2t} dt = \infty \]

In (2.4), this would force \( m = \infty \) which contradicts lemma 1. Thus, \( m(\rho) < 2/L \) for each \( \rho \in D \).

**Lemma 5.** Let \( C \) be a compact subset of \( D \). Then there exists a number \( \delta(C) > 0 \) such that \( L(\rho)m(\rho) \leq 2 - \delta \) for all \( \rho \in C \).

**Proof.** Suppose that the conclusion is not true. Then there are sequences \( \{\rho_n\} \) and \( \{\delta_n\} \) such that \( \delta_n > 0 \), \( \delta_n \rightarrow 0 \), \( \rho_n \rightarrow \rho_0 \in C \), and \( L(\rho_n)m(\rho_n) > 2 - \delta_n \).

The last inequality implies that \( L(\rho_n) \) and \( m(\rho_n) \) are positive. By lemma 4, it is true that \( 2 - \delta_n = L(\rho_n)m(\rho_n) < 2 \). Thus, \( \lim n \rightarrow \infty L(\rho_n)m(\rho_n) = 2 \). But by lemmas 2 and 3, we have that

\[ 2 = \lim_{n \rightarrow \infty} L(\rho_n)m(\rho_n) \leq \lim_{n \rightarrow \infty} L(\rho_n) \lim_{n \rightarrow \infty} m(\rho_n) = L(\rho_0)m(\rho_0) < 2 \]

which is a contradiction. Thus, there exists a \( \delta > 0 \) such that \( L(\rho)m(\rho) \leq 2 - \delta \) for all \( \rho \in C \).

3. THE MAIN RESULT.

We now show that the function \( m(\rho) \) is actually continuous on compact sets of the variable \( \rho \).

**Theorem.** Let \( C \) be a compact subset of \( D \). Then \( m(\rho) \) is continuous on \( C \).

**Proof.** Define \( h(t, \rho) = (d/dt)[f(\rho, u(t, \rho))] = f(u(t, \rho), u(t, \rho)) \hat{u}(t, \rho) \). Define

\( I(\rho) = \{ t \in [0, \omega) : h(t, \rho) < 2 - \frac{\bar{c} \delta}{2} \} \) where \( \delta \) is the number constructed in lemma 5.

Then \( I(\rho) \) contains an interval \((\tau(\rho), \omega(\rho))\) for some smallest \( \tau \in [0, \omega) \). For if \( m(\rho) > 0 \), then

\[ \lim_{t \rightarrow \tau^-} h(t, \rho) = \lim_{t \rightarrow \tau^-} f(\rho, u(t, \rho)) \lim_{t \rightarrow \tau^-} \hat{u}(t, \rho) = L(\rho)m(\rho) \leq 2 - \delta < 2 - \frac{\bar{c} \delta}{2} \]

If \( m(\rho) = 0 \), then \( \hat{u}(t, \rho) \) is positive and so \( \lim_{t \rightarrow \tau^-} f(\rho, u(t, \rho)) \) exists and

\[ \lim_{t \rightarrow \tau^-} h(t, \rho) = m(\rho) \lim_{t \rightarrow \tau^-} u(t, \rho) = 0 \]
If $m(p) < 0$, then $u(\omega, p) = \ell(\varepsilon)$ and
\[
\lim_{t \to \omega^-} h(t, \rho) = \lim_{u \to \ell} f(\varepsilon, u) m(p) \in [-\infty, 0] \tag{3.3}
\]

In all cases, there is a $\tau(p)$ such that $h(t, \rho) < 2 - \frac{1}{2} \delta$ on $(\tau, \omega)$ and $\tau$ is chosen as small as possible.

Let $\rho_0 \in C$ and suppose that $\tau_0 = \tau(\rho_0) > 0$. By the construction, $h(\tau_0, \rho_0) = 2 - \frac{1}{2} \delta$. But $h(\tau_0, \rho_0) = f_u(\varepsilon_0, u_0) \dot{u}_0 + f_{uu}(\varepsilon_0, u_0)^2$ where $u_0 = u(\tau_0, \rho_0)$, $\dot{u}_0 = \dot{u}(\tau_0, \rho_0)$, and $\ddot{u}_0 = \ddot{u}(\tau_0, \rho_0)$. Also, $2 - \frac{1}{2} \delta = f_u(\varepsilon_0, u_0) \ddot{u}_0$. Thus, $f_{uu}(\varepsilon_0, u_0) \leq 0$, $f_u(\varepsilon_0, u_0) > 0$, and $\ddot{u}_0 < 0$ imply that $h(\tau_0, \rho_0) < 0$. Consequently, $h(t, \rho_0) > 2 - \frac{1}{2} \delta$ on $[0, \tau_0)$. By the implicit function theorem, there exists a continuous function $t(\rho)$ and a number $\eta > 0$ such that $t(\rho_0) = \tau_0$ and $h(t(\rho), \rho) = 2 - \frac{1}{2} \delta$ for $|\rho - \rho_0| < \eta$. In fact, $t(\rho) = \tau(\rho)$ whenever $t(\rho) > 0$ (guaranteed by the uniqueness condition in the implicit function theorem). It follows immediately that the function, $\tau(p) = t(\rho)$ when $t(\rho) > 0$ and 0 otherwise, is continuous on $C$. Since $C$ is compact, $\tau^* = \sup\{t(\rho) : \rho \in C\}$ is finite.

Thus, $h(t, \rho) < 2 - \frac{1}{2} \delta$ for $t \geq \tau(p)$ since the $t$-derivative of $h$ is negative at a point where $h = 2 - \frac{1}{2} \delta$, and by the previous argument, $h(t, \rho) < 2 - \frac{1}{2} \delta$ for $t \geq \tau^*$. On the interval $[0, \tau^*]$, by continuous dependence of $u$ and by continuity of $f_u$, $f(\varepsilon, u(t, \rho)) \leq M = M(C)$. For $t \geq \tau^*$, $f_u(\varepsilon, u(t, \rho)) \dot{u}(t, \rho) < 2 - \frac{1}{2} \delta$ implies that
\[
f(\varepsilon, u(t, \rho)) \leq f(\varepsilon, u(\tau^*, \rho)) + (2 - \frac{1}{2} \delta)t \leq K + (2 - \frac{1}{2} \delta)t \tag{3.4}
\]
where $K$ is a uniform bound (again by continuous dependence of solutions $u$ on compact sets in the variable $(t, \rho)$).

In the equation (2.4) we had $m(\rho) = \beta - \int_0^\infty \lambda e^{-2t} \exp[f(\varepsilon, u(t, \rho))] dt$. Since the integrand is continuous on $[0, \infty) \times C$ and is uniformly bounded on the set $C$ by the integrable function $K \exp(-\delta t)$, $m(\rho)$ is a continuous function on $C$.

4. APPLICATIONS.

Consider the Dirichlet problem
\[
\Delta u + \lambda \exp[f(\varepsilon, u)] = 0, \quad x \in \Omega \tag{4.1}
\]
\[
u(x) = 0, \quad x \in \partial \Omega \tag{4.2}
\]
where $\Omega$ is the unit ball of $\mathbb{R}^2$ with center 0, and where $\Delta$ is the Laplace operator.

A typical example of a nonlinearity in applications (for catalysis problems) is $f(\varepsilon, u) = u/(1 + \varepsilon u)$. Using a result by Gidas, Ni, and Nirenberg [1], all solutions to (4.1)-(4.2) are radially symmetric; that is, $u = u(r)$ where $r = |x|$. Equations (4.1)-(4.2) then can be rewritten as
\[
u'' + \frac{1}{r} \nu' + \lambda \exp[f(\varepsilon, u)] = 0, \quad 0 < r < 1 \tag{4.3}
\]
\[
u'(0) = 0, \quad u(1) = 0 \tag{4.4}
\]
Making the change of variables $r = e^{-t}$, we have
\[
u + \lambda e^{-2t} \exp[f(\varepsilon, u)] = 0, \quad 0 < r < 1 \tag{4.5}
\]
Equation (4.5) with initial conditions \( u(0) = \alpha \) and \( \dot{u}(0) = \beta \) gives us equations (1.5)-(1.6). Let \( \epsilon = 0 \). Then \( f(0,u) = u \) and we have
\[
\dot{u} + \lambda e^{-2t} u = 0, \quad 0 < t < \infty
\]
\[
u(0) = \alpha, \quad \dot{u}(0) = \beta
\]
The solution to this is given by
\[
u(t,\lambda,\alpha,\beta,0) = \epsilon \left[ \frac{2k\lambda}{\lambda} \right] + (2-\nu\lambda) - 2 \epsilon \ln \left[ 1 + ke^{-t\sqrt{\lambda}} \right]
\]
where \( \lambda = (\beta-2)^2 + 2\lambda e^{\alpha} \) and \( k = \left( \nu\lambda + (\beta-2) \right) \left/ \left( \nu\lambda - (\beta-2) \right) \right. \). The boundary conditions \( u(0) = 0 \) and \( \dot{u}(\infty) = 0 \) imply that \( 2 - \sqrt{\lambda} = 0 \), or \( \lambda = \frac{1}{2}(4\beta - \beta^2) \). The bifurcation curve is given in figure 1.

A result by Dancer [2] shows the bifurcation curve to (4.3)-(4.4) is a 1-dimensional \( C^1 \)-manifold which is connected for each \( \epsilon \geq 0 \). The manifold has a boundary point at \((\lambda,\nu) = (0,0)\). In terms of the variables \((\lambda,\nu)\), the theorem shows that given a compact set \( C \) in \( D \) and a number \( \eta > 0 \) (but small), there is an interval \([0,\epsilon]\) contained in \([0,\epsilon]\) such that \( |m(\lambda,\beta,\epsilon) - m(\lambda,\beta,0)| < \eta \) whenever \((\lambda,\beta,\epsilon)\) is in the appropriate set. But \( m(\lambda,\beta,0) = 2 - \sqrt{\lambda} \), so
\[
2 - \sqrt{\lambda} - \eta < m(\lambda,\beta,\epsilon) < 2 - \sqrt{\lambda} + \eta
\]
In the region \( \{(\lambda,\beta): 2 - \sqrt{\lambda} + \eta < 0\} \), \( m(\lambda,\beta,\epsilon) \) is negative and in the region \( \{(\lambda,\beta): 2 - \sqrt{\lambda} - \eta > 0\} \), \( m(\lambda,\beta,\epsilon) \) is positive. The zeros of \( m \) must occur in the parabolic strip between these two regions. See figure 2.

5. OBSERVATIONS AND CONCLUSIONS.

The condition \( f(\epsilon,u) = u \) as \( \epsilon \to 0 \) was only needed to illustrate the example above. Similar results could be obtained if there is knowledge of a bifurcation result for other nonlinearities. For example, in Eberly [3], the nonlinearity \( u^\nu - 1 \) is analyzed with similar results, although there are an infinite number of branches of solutions to the condition \( \dot{u}(\infty) = 0 \).
The important condition used is that \( f_u(\varepsilon, u) + L(\varepsilon) \) as \( u \to \infty \). We conjecture that the condition \( f_u(\varepsilon, u) \geq 0 \) is technical and that the results on continuous dependence should hold for those nonlinearities \( \exp[f(u)] \) where \( f_{uu} \leq 0 \). For example, the nonlinearity \( g(\varepsilon, \kappa, \rho, u) = (1-\kappa \varepsilon u)^{\kappa} \exp[u/(1+\varepsilon u)] \), where \( \varepsilon, \kappa, \) and \( \rho \) are positive constants, also occurs in catalysis theory and this function has the property that \( (d^2/du^2)[\ln g(\varepsilon, \kappa, \rho, u)] \leq 0 \).

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