INCLUSIONS OF HARDY ORLICZ SPACES

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ABSTRACT. Let \( \phi \) be a continuous positive increasing function defined on \([0,\infty)\) such that \( \phi(x+y) \leq \phi(x) + \phi(y) \) and \( \phi(0) = 0 \). The Hardy-Orlicz space generated by \( \phi \) is denoted by \( H(\phi) \). In this paper, we prove that for \( \phi \neq \psi \), if \( H(\phi) = H(\psi) \) as sets, then \( H(\phi) \subseteq H(\psi) \) as topological vector spaces. Some other results are given.

KEY WORDS AND PHRASES. Modulus function, Orlicz spaces.

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1. INTRODUCTION.

Let \( \phi: [0,\infty) \to [0,\infty) \) be a continuous increasing function such that \( \phi(x+y) \leq \phi(x) + \phi(y) \) and \( \phi(0) = 0 \). Let \( T \) be the unit circle, and \( m \) be the Lebesgue measure on \( T \). A complex valued measurable function \( f \) defined on \( T \) is called \( \phi \)-integrable if \( \int_T |f(t)| \, dm(t) < \infty \). The space of all \( \phi \)-integrable functions on \( T \) will be denoted by \( L(\phi) \). This space was first introduced by Orlicz, [8]. Subsequent papers were written to study different aspects of \( L(\phi) \). Examples of these papers are Cater, [4], Gramsch, [5] and Pallaashke [9].

In [6] and [7], Lesniewicz introduced the so-called Hardy-Orlicz spaces \( H(\phi) \) for a given such function \( \phi \). The space \( H(\phi) \) was defined to be the space of all functions \( f \in L(\phi) \) such that \( f \) is the radial limit of some function \( g \) analytic in the open unit disc and belongs to the Nevalinna class \( N \). The relation between different \( H(\phi) \)-spaces was studied by Deeb, Khalil and Marzug [3]. In this paper, we show that the inclusion map between two \( H(\phi) \)-spaces is always continuous. Some other results are given. It should be remarked that in the work of Lesniewicz, [6], [7] and many other authors, \( \phi \) is assumed to be a \( \phi \)-convex function. In this paper it is not assumed so.

2. PRELIMINARIES AND NOTATIONS.

A function \( \phi: [0,\infty) \to [0,\infty) \) is called a modulus function if

1. \( \phi \) is continuous and increasing
2. \( \phi(x) = 0 \) if and only if \( x = 0 \)
3. \( \phi(x+y) \leq \phi(x) + \phi(y) \).

The functions \( \phi(x) = x^p, \ 0 < p \leq 1 \) and \( \phi(x) = \ln(1+x) \) are examples of modulus functions. Further, if \( \phi_1 \) and \( \phi_2 \) are modulus functions, then \( \phi_1 + \phi_2 \) and \( \phi_1 \circ \phi_2 \)
are modulus functions. Further, \( \psi = \frac{\phi}{1 + \phi} \) is a modulus function if \( \phi \) is.

Let \( T = \{ z : |z| = 1 \} \), \( \Delta = \{ z : |z| < 1 \} \). The space of analytic functions on \( \Delta \) is denoted by \( H(\Delta) \). Let \( H^p(\Delta) = \{ f \in H(\Delta) : \lim_{r \to 1^-} f(re^{i\theta}) \text{ exists a.e.} \} \). We will consider \( H^p(\Delta) \) as a space of functions on \( T \). For a given modulus function \( \phi \), we define:

\[
H(\phi) = \{ f \in H^p(\Delta) : \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|d\theta = \int_0^{2\pi} |\phi(f(e^{i\theta})|d\theta < \infty \}.
\]

The function \( d : H(\phi) \times H(\psi) \to [0, \infty) \), \( d(f, g) = \int_0^{2\pi} |\phi(f(e^{i\theta})) - \psi(g(e^{i\theta}))|d\theta \) defines a metric on \( H(\phi) \), under which \( H(\phi) \) becomes a topological vector space. If one assumes that \( \phi |u| \) is subharmonic for \( u \in H(\Delta) \), then \( H(\phi) \) turns out to be complete [2].

For \( f \in H(\phi) \), we write \( \|f\|_\phi = \int_T |f(e^{i\theta})|d\theta \leq \int_T |f(e^{i\theta})|d\theta \), \( H(\phi) = H^p \) and for \( \phi(x) = \ln(1 + x) \), \( H(\phi) = N^+ = \{ f \in N : \int_T \ln(1 + |f|) < \infty \} \), where \( N \) is the Nevalinna class.

3. \( H^1 \to H(\phi) \) IS CONTINUOUS.

In [2], it was shown that \( H^1 \subseteq H(\phi) \) for all modulus functions \( \phi \). The authors in [3] were not able to show that the inclusion map \( I : H^1 \to H(\phi) \) is continuous. In this section we prove that \( I : H^1 \to H(\phi) \) is continuous. Some other related questions are discussed.

THEOREM 2.1. Let \( \phi \) and \( \psi \) be two modulus functions such that \( \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lambda \) exists. Then:

(i) \( H(\phi) = H(\psi) \) if \( \lambda \neq 0 \) and \( \lambda \) is finite

(ii) \( H(\phi) \subseteq H(\psi) \) if \( \lambda = 0 \)

(iii) \( H(\psi) \subseteq H(\phi) \) if \( \lambda = \infty \).

PROOF. (i) Let \( \lambda \neq 0 \) be finite. Then there exists \( a_1, b_1, a_2, b_2 \in [0, \infty) \) such that

\[
\phi(x) \leq a_1 \psi(x) \quad \text{for} \quad x \in [a_2, \infty) \quad \text{(*)}
\]

\[
\psi(x) \leq b_1 \phi(x) \quad \text{for} \quad x \in [b_2, \infty) \quad \text{(**)}.
\]

Let \( f \in H(\phi) \). Set \( E(a_2) = \{ t \in T : |f(t)| \geq a_2 \} \). Then

\[
\|f\|_\phi = \int_{E(a_2)} \phi(f(e^{i\theta}))d\theta + \int_{E^c(a_2)} \psi(f(e^{i\theta}))d\theta
\]

\[
\leq a_1 \|f\|_\psi + \phi(a_2) < \infty.
\]

Hence \( f \in H(\phi) \) and \( H(\phi) \subseteq H(\phi) \). Similarly we show \( H(\phi) \subseteq H(\psi) \). Consequently, \( H(\phi) = H(\psi) \). Case (ii) and (iii) are proved similarly and details are omitted. This ends the proof.

THEOREM 2.2. Let \( \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lambda > 0 \). Then the inclusion map \( I : H(\phi) \to H(\psi) \) is continuous.

PROOF. From the proof of Theorem 2.1, there exists \( a, b > 0 \) such that \( \|f\|_\psi \leq \psi(a) \ast b \|f\|_\phi \) for all \( f \in H(\phi) \).

Let \( f_n \to 0 \) in \( H(\phi) \). Thus the sequence \( (f_n) \) is bounded in the metric of \( H(\phi) \) and consequently bounded in \( H(\psi) \). If possible let there exist a subsequence \( (f_{n_k}) \)
such that \( \|f_{n_k}\| \to a > 0 \). Since \( \|f_{n_k}\| \to 0 \), \((f_{n_k})\) has a subsequence which converges pointwise to the zero function. With no loss of generality, we can assume that \( f_{n_k} \to 0 \) a.e. Another application of the proof of Theorem 2.1, yields \( \psi(x) \leq \psi(a) + b \cdot \phi(x) \) for all \( x \in [0, \infty) \). Hence
\[
\psi \left( \frac{f_{n_k}(t)}{n_k} \right) \leq \psi(a) + b \cdot \phi \left( \frac{f_{n_k}(t)}{n_k} \right).
\]
The sequence of functions \( g_{n_k}(a) = \psi(a) + b \cdot \phi \left( \frac{f_{n_k}(t)}{n_k} \right) \) converges a.e. to \( \psi(a) \) and
\[
\int_T g_{n_k}(t) \, dt \to \psi(a).
\]
Consequently, by the generalized Lebesgue convergence theorem, [10], we have
\[
\lim_{n_k} \int_T \frac{f_{n_k}(t)}{n_k} \, dt = \int_T \frac{f(t)}{n_k} \, dt \to 0.
\]
This is a contradiction. Thus, the point \( w = 0 \) is the only limit point of the bounded sequence \( (\|f_n\|) \). Consequently, [11], the sequence \( \|f_n\| \) converges to zero. Hence \( I: H(\phi) \to H(\psi) \) is continuous. This ends the proof.

**Corollary 2.3.** If \( \lim_{x \to \infty} f(x) = \lambda \in (0, \infty) \), then \( H(\phi) \) is topological vector spaces.

**Proof.** By Theorem 2.1, \( H(\phi) = H(\psi) \) as sets. Theorem 2.2 implies that \( I: H(\phi) \to H(\psi) \) is an isomorphism. This ends the proof.

A linear map \( A: H(\phi) \to H(\psi) \) is called metrically bounded if \( \|Af\|_{\psi} \leq \lambda \|f\|_{\phi} \) for all \( f \in H(\phi) \) and some \( \lambda > 0 \). Clearly every metrically bounded map is continuous. The converse need not be true. However, for the inclusion map, we have the following:

**Theorem 2.4.** Let \( \phi \) be any modulus function. Then there exists \( \lambda > 0 \) such that for all \( f \in H^1 \), \( \|f\|_1 \geq 1 \), \( \|f\|_{\phi} \leq \lambda \|f\|_1 \).

**Proof.** It is known, [2] (and easy to show) that \( H^1 \subseteq H(\phi) \) for all modulus functions \( \phi \). If \( f \in H^1 \) and \( \|f\|_1 = 1 \), then using the argument in Theorem 2.1, we have \( \|f\|_{\phi} \leq \lambda \|f\|_1 \).

Let \( f \in H^1 \), \( \|f\|_1 > 1 \). Then there exists \( 0 < \alpha < 1 \) such that \( \|af\|_1 = 1 \). Since \( \alpha < 1 \), there exists a natural number \( n \) such that \( \frac{1}{n+1} \leq \alpha \leq \frac{1}{n} \). Hence
\[
\|af\|_{\phi} \leq \lambda \|af\|_1 = \lambda \alpha \|f\|_1.
\]
But \( \|\frac{1}{n+1} \|f\|_{\phi} \leq \|af\|_{\phi} \), and \( \|\frac{1}{k} \|f\|_{\phi} \geq \|\frac{1}{k} \|f\|_{\phi} \) for any modulus function \( \phi \). It follows that:
\[
\frac{1}{n+1} \|f\|_{\phi} \leq \lambda \alpha \|f\|_1 \leq \frac{\lambda}{n} \|f\|_1,
\]
and consequently
\[
\|f\|_{\phi} \leq \lambda \frac{n+1}{n} \|f\|_1 \leq 2\lambda \|f\|_1.
\]
This ends the proof.

**Theorem 2.5.** Let \( \phi \) be a given modulus function such that \( H^1 = H(\phi) \). If metric and topological bounded sets coincide in \( H(\phi) \), then \( \|f\|_{\phi} \leq \lambda \|f\|_1 \) for all \( f \in H(\phi) \), \( \|f\|_{\phi} \leq 1 \) for some \( \lambda > 0 \).
PROOF. Applying Corollary 2.3, \( I: H(\phi) \rightarrow H^1 \) is an isomorphism of topological vector spaces. If possible, let \( \| f \|_1 < \lambda \| f \|_{\phi} \) be not true on the unit sphere of \( H(\phi) \). Then, for each \( n \), there exists \( f_n \in H(\phi) \), \( \| f_n \|_{\phi} = 1 \) such that

\[
\| f_n \|_1 > n \| f_n \|_{\phi} = n
\]

Consider the sequence \( \frac{f_n}{n} = g_n \). By the assumption on bounded sets of \( H(\phi) \), we have, \( [12] \), \( g_n \rightarrow 0 \) in \( H(\phi) \). But \( \| g_n \|_1 = \| \frac{f_n}{n} \|_1 \geq 1 \) for all \( n \). This contradicts the continuity of the identity map \( I: H(\phi) \rightarrow H^1 \). Hence there exists \( \lambda > 0 \) such that:

\[
\| f \|_1 \leq \lambda \| f \|_{\phi} \quad (*)
\]

for all \( f \in H(\phi) \), \( \| f \|_{\phi} = 1 \).

Let \( f \in H(\phi) \), \( \| f \|_{\phi} < 1 \). Consider the map \( K: [0, \infty) \rightarrow [0, \infty) \), \( K(t) = \| tf \|_{\phi} \). It can be easily seen that \( K \) is continuous. Hence there exists \( a > 1 \) such that \( K(a) = 1 \). Thus for every \( f \in H(\phi) \), \( \| f \|_{\phi} < 1 \), we can find \( a > 1 \) such that \( \| af \|_{\phi} = 1 \). Hence, from equation (\( * \)), we get:

\[
\| af \|_1 \leq \lambda \| af \|_{\phi} \leq 2a \| f \|_{\phi}.
\]

Consequently, \( \| f \|_1 \leq 2 \| f \|_{\phi} \). This ends the proof.

4. FURTHER RESULTS

The concept of metrically bounded linear operator was introduced in Section 3. A linear map \( A: H(\phi) \rightarrow H(\psi) \) is called metrically bounded if there exists \( \lambda \in (0, \infty) \) such that \( \| Af \|_{\psi} \leq \lambda \| f \|_{\phi} \). In general, a continuous linear map need not be metrically bounded. In this section we prove a result which is a generalization of Theorem 3.1 in [3].

THEOREM 4.1. Let \( \phi \) and \( \psi \) be any two modules functions. Then the following are equivalent:

(i) \( \lim_{x \to 0} \frac{\phi(x)}{\psi(x)} = \delta \) \( \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \varepsilon \), for some \( \varepsilon, \delta \in (0, \infty) \).

(ii) \( H(\phi) = H(\psi) \), and the identity map \( I \) is metrically bounded.

PROOF. (i) \( \rightarrow \) (ii). From the assumption in (i), one can choose \( a \) and \( b \) in \( (0, \infty) \) such that

\[
\frac{\phi(x)}{\psi(x)} > r \quad \text{on} \quad [0, a]
\]

\[
\frac{\phi(x)}{\psi(x)} > s \quad \text{on} \quad (b, \infty)
\]

for some \( r, s \in (0, \infty) \). Theorem 3.2 implies that \( H(\phi) = H(\psi) \).

Let \( f \in H(\phi) \). Consider the following sets:

\[
\{ x \in (0, \infty) : \quad \frac{\phi(x)}{\psi(x)} > r \} \quad \text{and} \quad \{ x \in (0, \infty) : \quad \frac{\phi(x)}{\psi(x)} > s \}
\]
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\[ E(a) = \{ t: 0 \leq \| f(e^{it}) \| < a \} \]
\[ E(b) = \{ t: \| f(e^{it}) \| > b \} \]
\[ E(a,b) = \{ t: a \leq \| f(e^{it}) \| \leq b \} \]

Then:

\[ \| f \|_\psi = \int_{E(a)} \psi \| f(e^{it}) \| dt + \int_{E(a,b)} \psi \| f(e^{it}) \| dt + \int_{E(b)} \psi \| f(e^{it}) \| dt \leq \frac{1}{s} \| f \|_\phi + \int_{E(a,b)} \psi \| f(e^{it}) \| dt + \frac{1}{s} \| f \|_\phi . \]

On the closed interval \([a,b]\), the continuity of \( \frac{\phi(x)}{\psi(x)} \) implies the existence of \( \lambda > 0 \) such that \( \psi(x) \leq \lambda \phi(x) \). Hence

\[ \int_{E(a,b)} \psi \| f(e^{it}) \| dt \leq \frac{1}{\lambda} \| f \|_\phi . \]

Thus, \( \| f \|_\psi \leq \beta \| f \|_\phi \) where \( \beta = \max\left( \frac{1}{s}, \frac{1}{\lambda} \right) \). In a similar way one can show that \( \| f \|_\phi \leq \gamma \| f \|_\psi \) for all \( f \in H(\phi) = H(\psi) \). Hence the identity map is metrically bounded.

Conversely, (ii) \(\Rightarrow\) (i). Assume \( H(\phi) = H(\psi) \) and \( I: H(\phi) \rightarrow H(\psi) \) is metrically bounded. Then there exists \( \alpha \) and \( \beta \) in \((0,\infty)\) such that

\[ \| f \|_\phi \leq \alpha \| f \|_\psi \leq \| f \|_\phi . \]

Hence

\[ \frac{\alpha}{\beta} \leq \frac{\| f \|_\phi}{\| f \|_\psi} \leq \alpha \text{ for all } f \in H(\phi) = H(\psi) . \]

Consider the function \( f(z) = xz \) for \( z = e^{it} , x \in (0,\infty) \). Then

\[ \| f \|_\phi = \phi(x) \text{ and } \| f \|_\psi = \psi(x) . \]

Consequently

\[ \frac{\alpha}{\beta} \leq \frac{\phi(x)}{\psi(x)} \leq \alpha . \]

Since \( \alpha, \beta \in (0,\infty) \), (i) then follows. This end the proof.

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