ON THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

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ABSTRACT. This paper is concerned with some general theorems on the distributional Stieltjes transformation. Some Abelian theorems are proved.

KEY WORDS AND PHRASES. Stieltjes transforms of distributional, asymptotic behavior, Abelian theorems.

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1. REGULARLY VARYING FUNCTIONS

Throughout the paper, r will denote a positive continuous function on an interval \((X, \infty)\), \(X > 0\), such that the limit

\[
\lim_{t \to \infty} \frac{r(pt)}{r(t)}
\]

exists for every \(p > 0\). Such functions are called regularly varying functions (r.v.f.) at infinity and it is well known ([7]) that they are of the form \(r(t) = t^a L(t)\) for some \(a \in \mathbb{R}\) (called the order or index of \(r\)) and some slowly varying function (s.v.f.) \(L\). This means that the function \(L : (X, \infty) \to (0, \infty)\) is continuous and that

\[
\lim_{t \to \infty} \frac{L(pt)}{L(t)} = 1
\]

for every \(p > 0\).

2. QUASIASYMPTOTIC BEHAVIOUR AT INFINITY RELATED TO \(r\)

The quasiasymptotic behaviour (q.a.b.) at infinity of tempered distributions with support in \([0, \infty)\) (denoted by \(S_\infty^+\)) was defined by Zavjalov (see, for instance, [2]). In this paper we use a somewhat more general concept of q.a.b., related to a r.v.f. as defined and analysed in [10].

Definition 1. Let \(T \in S_\infty^+\) and \(r\) be some r.v.f. . The distribution \(T\) has q.a.b. at infinity related to \(r\) if there exists the limit in the sense of \(S_\infty^+\):

\[
\lim_{t \to \infty} \frac{T(kt)}{r(k)} = g(t)
\]

provided that \(g \neq 0\).
If the order of \( r \) is \( a \), then \( g(t) = A f_{a+1}(t) \) for some \( A \neq 0 \) (from now on we take \( A = 1 \) for simplicity), where

\[
f_{a+1}(t) = H(t) t^a / \Gamma(a+1) \quad \text{for } a \geq 0 \quad \text{and} \quad f_{a+1}(t) = D^n f_{a+n+1}(t)
\]

for \( a < 0 \) and \( n+a > 0, n \in \mathbb{N} \). As usual, \( H \) is the characteristic function of the interval \((0,\infty)\), and \( D \) stands for the distributional derivative.

It is easy to see that a continuous function on \([0,\infty)\) having ordinary asymptotic behaviour of order \( a \geq -1 \) related to \( r \) has also q.a.b. of the same order and conversely. However for \( a < -1 \) this may not be true. This follows from the following

**Structural Theorem.** ([10]) A distribution \( T \in \mathcal{S}' \) has q.a.b. at infinity related to a r.v.f. \( r \) of order at iff there exist a natural number \( n \), \( n+a > 0 \), and a continuous function \( F \) on \( \mathbb{R} \) such that

\[
T \ast f \quad \text{and} \quad F(t) \sim \frac{1}{\Gamma(n+a+1)} t^n r(t) \quad \text{as} \quad t \to \infty.
\]

The proof of this important theorem is analogous to the one of Theorem I in [2], p. 373.

3. **EQUIVALENCE AT INFINITY**

The other "asymptotic behaviour" of distributions at infinity given in the following definition was used in [3], [1] and [6]; however, this notion goes back to Sebastiao e Silva ([8]).

**Definition 2.** A distribution \( T \in \mathcal{S}' \) is equivalent at infinity to \( r(t) = t^a L(t) \), \( a \in \mathbb{Z}_- \), if for some \( X' \), \( X' \geq X \), and some nonnegative integer \( n \), \( n+a > 0 \), there exists a continuous function \( F \) on \([X',\infty)\) such that \( T \ast n F \) on \([X',\infty)\) and

\[
F(t) \sim t^n r(t)/(a+1)(a+2)\ldots(a+n)
\]

in the ordinary sense as \( t \to \infty \).

It seems to be of interest to compare these two asymptotics; for our purposes it is enough to prove

**Lemma 1.** Let \( T \in \mathcal{S}' \) be equivalent at infinity to \( r(t) = t^a L(t) \) for \( a > -1 \). Then it has q.a.b. of order \( a \) related to \( r \).

**Proof.** We can write \( T = B + B^n F(t) \), where the supports of \( B \) and \( F \) are, respectively, in \([0,X')\) and \([X',\infty)\), \( X' > 1 \). Let us prove that

\[
\lim_{k \to \infty} \frac{B(kt)}{k^a L(k)} = 0
\]

In fact, for every \( \varepsilon > 0 \) there exists a number \( n_1 \in \mathbb{N}_0 \) (\( N_0 = \mathbb{N} \cup \{0\} \)) and a continuous function \( F_1 \) on \( \mathbb{R} \) such that \( D^{n_1} F_1 = B \) and supp \( F_1 \subset [-\varepsilon, X' + \varepsilon] \). For \( \phi \in \mathcal{S} \) we have

\[
\langle \frac{B(kt)}{k^a L(k)}, \phi(t) \rangle = \int_{-\varepsilon}^{X' + \varepsilon} \frac{F_1(t)}{k^{n_1+a+1} L(k)} (n_1)(t/k) dt \to 0
\]

(3.1)
since $n_1+a+1 > a+1 > 0$. By supposition $F$ satisfies (3.1), so by the Structural theorem it has q.a.b. of order $a$ related to $r$.

4. STIELTJES TRANSFORM OF DISTRIBUTIONS

For the sake of completeness we rewrite the definition of the distributional Stieltjes transform given in [4]. Let $I'(z), z \in \mathbb{C}$, denote the subspace of distributions $t \in \mathcal{S}_+$ such that $T = D^n G$ for some $n \in \mathbb{N}$ and some locally integrable function $G$ on $\mathbb{R}$ with support in $[0, \infty)$ and

$$\int |G(t)| t^{-(z+n+1)} dt < \infty.$$ 

From now on we take $z \in \mathbb{R}$ and $z > -1$, though a complex setting is also possible (see [4] or [1]). Obviously $I'(z) \subset \mathcal{S}_+$ and $I'(z_1) \subset I'(z_2)$ for $-1 < z_1 < z_2$.

**Definition 3.** The Stieltjes transform of index $z$ of a distribution $T \in I'(z)$ is the complex valued function

$$S_z(T)(s) = \langle T(t), \frac{h(t)}{(t+s)^{z+1}} \rangle, \quad s \in \mathbb{C} \setminus (-\infty, 0],$$

where $h$ is an infinitely differentiable function on $\mathbb{R}$ such that $h(t) = 1$ in some neighbourhood of $[0, \infty)$ and $h(t) = 0$ in some interval $(-\infty, -\varepsilon)$, $\varepsilon > 0$.

It is easy to see that (4.1) does not depend on the function $h$, so it is usually omitted. It is proved in [5] that $S_z(T)(s)$ is a holomorphic function of the complex variable $s$ in the domain $\mathbb{C} \setminus (-\infty, 0]$ provided that $T \in I'(z)$. We shall need the following equality ([5], p. 140)

$$S_{z+n}(T)(s) = \frac{1}{(z+1)(z+2)\ldots(z+n)} S_z(D^n T)(s)$$

for $T \in I'(z)$ and $n \in \mathbb{N}$. Observe that $T \in I'(z)$ implies $T \in I'(z+n)$ and $D^n T \in I'(z)$.

5: ABELIAN THEOREMS

The initial value type Abelian theorems for the distributional Stieltjes transform seem to have a satisfactory form. So, we prove only final value type ones. We use first the following result from [6]:

**Theorem 1.** Let us suppose that $T \in I'(z)$ is equivalent at infinity to a regularly varying function $r(t) \sim t^a L(t)$ of order $a > -1$. Then

$$S_z(T)(s) \sim B(a+1, z-a) L(s) s^{a-z} \quad \text{as} \quad s \rightarrow \infty, \quad s \in \mathbb{R},$$

provided that $z > a > -1$.

As usual, $B(p, q)$ stands for the beta function. In view of Lemma 1 we see that this Theorem can be rewritten as

**Theorem 1'.** Let us suppose that $T \in \mathcal{S}_+$ has q.a.b. of order $a > -1$ related to the r.v.f. $r(t) = t^a L(t)$. Then (5.1) holds if $z > a > -1$.

If $T$ in these two theorems is a continuous function on $[0, \infty)$, then $T(t) \sim t^a L(t)$ as $t \rightarrow \infty$ in the ordinary sense. Essentially, we need such
a "functional" (i.e. not "distributional") version of them in the following

**Abelian Theorem.** Let $T \in \mathcal{S}'$ have q.a.b. of order $a$ related to a r.v.f. $r(t) = t^a L(t)$. Then

i) $T \in \mathcal{I}'(z)$ for $z > \max(-1,a)$

and

ii) $S_z(T)(s) \sim \frac{\Gamma(z-a)}{(z+1)!} L(s) s^{a-z}$ as $s \to \infty$, staying on the real line.

**Remark.** Such a statement was proved in [3] for $r(t) = t^a$, $a > -1$ and in [4] for $r(t) = t^a \log^{\frac{1}{a}} t$, $a > -1$. Further on, r.v.f. were used in [6] (again for $a > -1$). In all these papers the equivalence at infinity was used. The q.a.b. was used in [9] for $r(t) = a^a$ ($a$ - arbitrary real number) and now for any r.v.f. In [1] the results from [3] are given in a complex setting; it might be of interest to prove an analogous statement for our Abelian theorem.

**Proof of the Abelian theorem.** Part i) follows from the Structural theorem and the estimate $L(t) < C t^\varepsilon$ for $t \geq t_0(\varepsilon)$ ($\varepsilon$ in $(0,1)$).

For ii), we take $n > -a$ and $F$ as in the structural theorem; then

$$F(t) \sim C_n t^{n+a} L(t) \text{ as } t \to \infty$$

for $C_n = 1/\Gamma(n+a+1)$. By Theorem 1' we get

$$S_{z+n}(F)(s) \sim C_n B(n+a+1,z+n-(n+a)) L(s) s^{a-z} \text{ as } s \to \infty,$$

and from (4.2) we have

$$S_z(T)(s) \sim (z+1)(z+2)\ldots(z+n) S_{z+n}(F)(s), \text{ so}$$

$$S_z(T)(s) \sim C_n \frac{\Gamma(n+a+1) \Gamma(z-a)}{\Gamma(z+1)} L(s) s^{a-z}. \tag{5.2}$$

This gives the statement ii).

**Example.** The equivalence at infinity with the distribution

$$T = A(a,j) F_p(t^a \log^{\frac{1}{a}} t), \quad a \in \mathbb{R}, \quad j \in \mathbb{N}_0 \tag{5.3}$$

for appropriate constant $A(a,j)$ was analysed in [4]: $F_p$ stands for the finite part. Obviously, $T$ is equivalent at infinity to $t^a \log^{\frac{1}{a}} t$ for $a \in \mathbb{Z}$. We take $A(a,j) = 1$ then. On the other hand, $T$ has q.a.b. of order $a$ related to $t^a \log^{\frac{1}{a}} t$ for $a \notin \mathbb{Z}$ and related to $A(a,j)t^a \log^{j+1} t$ for $a \in \mathbb{Z}$; we take $A(a,j) = (-1)^{-a-1}/((-a-1)!(j+1))$ then. Computing the Stieltjes transform of $T$ we see that it behaves at infinity as the Abelian theorem predicts (see [4], formulae (2.3) and (2.4)).

Now let $-2 < a < -1$. Then the distribution $S = T + \delta$ has q.a.b. of order $-1$ related to $1/t$ and is equivalent at infinity with $t^a \log^{\frac{1}{a}} t$. But for $z > -1$

$$S_z(S) = S_z(T) + S_z(\delta) \sim C_{a,j} \log^{j+1} s s^{a-z} + s^{-\log^{j+1} s} \sim s^{-\log^{j+1} s} \text{ as } s \to \infty.$$

This trivial example (which can be generalized easily) shows
again that q.a.b. is more appropriate for final value type Abelian theo-
rems for Stieltjes transformation than equivalence at infinity, though
the latter seems more "natural".

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