A NOTE ON THE INVERSE FUNCTION THEOREM
OF NASH AND MOSER

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ABSTRACT. The Nash-Moser inverse function theorem is proved for different kind of
differentiabilities.

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smoothing operators, tame map, \( C^\infty_\alpha \)-differentiability.

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1. INTRODUCTION

The purpose of this note is to formulate the inverse function theorem of Nash and
Moser for different differentiabilities using a categorical approach. The proof is
based on the inverse function theorem of Nash and Moser in the version of Hamilton [1]
formulated in the category of graded Frechet spaces which admit smoothing operators and
C\(\infty_\alpha \)-differentiable [2] tame maps. Our proof is using the same technique as Schmid [3]
uses when he proves this theorem for a stronger notion of differentiability, called
the \( \Gamma \)-differentiability, than the \( C^\infty_\alpha \)-differentiability. From our formulation it is
possible to derive the inverse function theorem of Nash and Moser for natural
differentiabilities stronger than the \( C^\infty_\alpha \)-differentiability.

2. THE INVERSE FUNCTION THEOREM OF NASH AND MOSER

Let \( LC \) denote the category of locally convex limit vector spaces [2] and con-
tinuous linear mappings. Further let \( K_\alpha \) denote a coreflective subcategory of \( LC \)
which is closed under finite products and the coreflector \( \gamma^\alpha : LC \rightarrow K_\alpha \) is the identity
on morphisms and such that the identity mapping \( (C^\infty_\alpha (X,F))^\alpha = C^\infty_\alpha (X,F) \rightarrow C^\infty_\alpha (X,F) \) is
continuous. Here \( C^\infty_\alpha (X,F) \) denotes the vector space of continuous mappings \( X \rightarrow F \)
endowed with continuous convergence [2], and \( X \) is a limit space and \( F \in \text{obj}(LC) \).

For any pair \( E,F \in \text{obj}(LC) \) we let \( L^k_E(F) \) be the space of all continuous
\( k \)-linear mappings from \( E^k \) into \( F \), endowed with continuous convergence. We write
\( (L^k_E(F))^\alpha = L^k_{\alpha E}(F) \).

DEFINITION. Let \( E \) and \( F \) be locally convex spaces and let \( U \) be open in \( E \).
A mapping $f : U \to F$ is said to be differentiable of class $C^p_\alpha$, if there exist functions

$$b^k f : U \to L^k(E,F), \quad k = 0, 1, \ldots, p,$$

such that $D^0 f = f$ and for each $x \in U$, each $h \in E$ and each $k = 0, 1, \ldots, p-1$, we have

$$\lim_{t \to 0} t^{-1} (b^k f(x+th) - b^k f(x)) = b^{k+1} f(x) h,$$

and such that for each $k \in \mathbb{N}$, $k \leq p$, the following two conditions are satisfied:

1. $D^k f(U) \subseteq L^k(E,F)$
2. $D^k f : U \to L^k(E,F)$ is continuous.

$f$ is called differentiable of class $C^\infty_\alpha$ if it is differentiable of class $C^p_\alpha$ for every $p \in \mathbb{N}$.

By Keller [2] the chain rule is valid for $C^\infty_\alpha$, since $\alpha$ is a finer limit structure than continuous convergence. From the universal property of continuous convergence follows that for any continuous map $g : U \to L^k(E,F)$ the associated map $\hat{g} : U \times E^k \to F$ defined by $\hat{g}(x,h_1,\ldots,h_k) = g(x)(h_1,\ldots,h_k)$, $x \in U$, $h_i \in E$, is continuous. As the limit structure $\alpha$ is always finer than $\alpha$, we have that differentiability of class $C^\infty_\alpha$ implies differentiability of class $C^\infty_\alpha$. The latter is exactly the concept of differentiability used by Hamilton [1] to prove the inverse function theorem of Nash and Moser.

We first recall some definitions that will be needed.

Let $E$ be a Fréchet space. A grading on $E$ is an increasing sequence of norms $(\| \cdot \|_r)_{r \in \mathbb{N}}$ on $E$ which defines the topology on $E$. Two gradings $(\| \cdot \|_r^1)_{r \in \mathbb{N}}$ and $(\| \cdot \|_r^2)_{r \in \mathbb{N}}$ are equivalent if for some $s \in \mathbb{N}$ $\| x \|_r^1 \leq c \| x \|_r^2$ and $\| x \|_r^2 \leq c \| x \|_r^{2+s}$, $x \in E$, with a constant $c$ which may depend on $r$. A graded space is a Fréchet space together with an equivalence class of gradings. We say that a graded space $E$ admits smoothing operators if we can find linear maps $S_t : E \to E$, $1 \leq t < \infty$, such that for some $r$ $\| S_t(x) \|_{i+k} \leq c t^{r-k} \| x \|_i$ and $\| S_t(x) - x \|_i \leq c t^{r-k} \| x \|_{i+k}$ for all $i,k \in \mathbb{N}$, $1 \leq t < \infty$, $x \in E$ and some constant $c$ which may depend on $i$ and $k$.

Let $E$ and $F$ be graded spaces and $U$ open in $E$. We say that a map $f : U \to F$ is tame if for every $x_0 \in U$ we can find a neighbourhood $U_0$ and a number $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have the growth estimate $\| f(x) \|_n \leq c (\| x \|_{n+r} + 1)$ for all $x_0 \in U$, where the constant $c$ may depend on $n$.

In the proof of the inverse function theorem of Nash and Moser we shall also need the following result (Lemma 2, [3]): The composition of two continuous tame maps is continuous and tame.

**Theorem.** Let $E$ and $F$ be graded spaces which admit smoothing operators. Let $U$ be open in $E$ and assume that

1. $f : U \to F$ is differentiable of class $C^\infty_\alpha$ and tame.
2. $D^k f : U \times E^k \to F$ is tame for every $k \in \mathbb{N}$.
3. For each $x \in U$ the derivative $Df(x) : E \to F$ is an isomorphism.
4. The map $Vf : U \to L^\infty_\alpha(F,E)$, $Vf(x) = (Df(x))^{-1}$, is continuous.
5. $Vf : U \times F \to E$ is tame.
Then for any \( x_0 \in U \) we can find open neighbourhoods of \( x_0 \) and \( V_0 \) of \( f(x_0) \) such that \( f \) is a bijective map from \( U_0 \) onto \( V_0 \) and the inverse map \( f^{-1} : V_0 \to U_0 \) is differentiable of class \( C_\alpha \) and the maps \( D^k f^{-1} : V_0 \times F^k \to E \) are tame for all \( k \in \mathbb{N} \). Furthermore we have the formula \( D(f^{-1})(y) = Vf(f^{-1}(y)) \) for all \( y \in V_0 \).

**Proof.** The maps \( D^k f : U \times E^k \to F \) are continuous and tame, since \( f \) is differentiable of class \( C_\alpha \) and assumption (2). Further the assumptions (4) and (5) imply that also \( Vf : U \times F \to E \) is continuous and tame. Now we have that \( f \) is differentiable of class \( C_\alpha \) and all \( D^k f \) are tame, \( Df(x) : E \to F \) is an isomorphism for every \( x \in U \) and the family of inverses \( Vf : U \times F \to E \) are continuous and tame maps. Consequently the conditions of the inverse function theorem of Nash–Moser are fulfilled (Theorem 1.1.1 p. 171 in [1]). Then for every \( x_0 \in U \) there exist neighbourhoods \( U_0 \) of \( x_0 \) and \( V_0 \) of \( f(x_0) \) such that \( f : U_0 \to V_0 \) is bijective and \( f^{-1} : V_0 \to U_0 \) is continuous and tame. Furthermore the formula \( \lim_{t \to 0} (f^{-1}(y + tw) - f^{-1}(y)) = Vf(f^{-1}(y))w \) holds, for all \( y \in V_0 \) and \( w \in F \), by the proof of Theorem 1.1.1 p. 186 in [1]. By induction on \( k \) we will prove the remaining part that \( f^{-1} : V_0 \to U_0 \) is differentiable of class \( C_\alpha \) and \( D^{k+1} f^{-1} : V_0 \times F^k \to E \) is tame for every \( k \in \mathbb{N} \). From the formula \( Df^{-1} = Vf \cdot f^{-1} \) and assumption (4) follow that \( Df^{-1} : V_0 \to L(F,E) \) is continuous. Further we have that \( D^{k+1} f^{-1} : V_0 \times F \to E \) is tame since \( Vf \) is continuous and tame. Assume now it to be true for \( k \). From the definition of the \( C_\alpha \)-differentiability follows that the map \( f^{-1} \) is \( C_{\alpha+k+1} \) if \( Df^{-1} \) is differentiable of class \( C_\alpha \). Since \( Df^{-1} = Vf \cdot f^{-1} \), \( D^{k+1} f^{-1} \) is clearly tame so we only have to show that \( Vf \) is differentiable of class \( C_\alpha \). By induction on \( p \). By Theorem 5.3.1, p. 102 in [1] we have that \( Vf \) is weakly differentiable and that \( D(Vf) : U_0 \times E \times F \to E \) is continuous and the formula \( [D(Vf)](x)(u,w) = -Vf(x)[D^2f(x)(u,Vf(x)w)] \) holds for all \( x \in U_0 \), \( u \in E \) and \( w \in F \). Thus the derivative \( D(Vf) : U_0 \to L_\alpha(E \times F,E) \) can be factorized according to

\[
U_0 \xrightarrow{Df^{-1}} L_\alpha(E,F) \times L_\alpha(F,E) \xrightarrow{h} L_\alpha(E \times F,E)
\]

where \( h \) is defined by \( h(\phi,\psi) = -\psi \cdot \phi \cdot (id_E,\psi) \) for \( \phi = D^2f(x) \) and \( \psi = Vf(x) \). By Theorem 0.3.5 in [2] \( h \) is continuous for \( \alpha = \infty \). Since the category \( K_\alpha \) is closed under finite products and \( ?^\alpha \) is a coreflector it follows that \( h \) is continuous. Thus it is true for \( p = 0 \). Since \( h \) is bilinear it is differentiable of class \( C_\alpha \), and consequently the map \( Vf \) is differentiable of class \( C_\alpha \) by induction. Thus the theorem is proved.

We shall now consider examples of coreflective subcategories of LC which are closed under finite products and the coreflectors \( ?^\alpha \) fulfill the assumption that the identity mapping \( C_\alpha(U,F) \to C_c(U,F) \) is continuous.

**Example 1.** Let \( K_\alpha \) be the category \( K_c = LC \); \( ?^\alpha \) is the identity functor \( 1_{KC} = ?^\alpha \).

**Example 2.** Let \( K_\alpha \) be the category \( K_e \) of equable locally convex limit vector spaces [2]. The coreflector \( ?^e : LC \to KE \) is the identity on morphisms and on objects \( E \) it is characterized as follows: a filter \( F \) on \( E \) converges to zero in \( E^e \) iff \( \forall G \leq F \) for some filter \( G \) which converges to zero in \( E \).
EXAMPLE 3. Let $K$ be the category $K_M$ of Marinescu spaces [2]. The coreflector $\Gamma^M: LC \to K_M$ is the identity on morphisms and on objects $E$ it is characterized as follows: a filter $F$ on $E$ converges to zero in $E^M$ iff $\forall G = G \leq F$ and $\cap \{ G \subseteq G : G \in G \} \in G$ for some filter $G$ which converges to zero in $E$.

EXAMPLE 4. Let $K_b$ be the category $K_b$ of bornological locally convex limit vector spaces. The coreflector $\Gamma^b: LC \to K_b$ is the identity on morphisms and on objects $E$ it is characterized as follows: a filter $F$ on $E$ converges to zero in $E^b$ iff $\forall B \subseteq F$ for some bounded subset $B \subseteq E$, i.e. some set $B$ such that $\forall B$ converges to zero in $E$.

Example 1 gives us the inverse function theorem of Nash and Moser by Hamilton [1]. From example 3 we derive the inverse function theorem of Nash and Moser for the differentiability of class $C^\infty_M$ ($C^\infty_B$ in Keller [2]). In [4] Kriegl has discussed smooth mappings between locally convex spaces, where a mapping is called smooth iff its composition with smooth curves are smooth. He has compared this concept of smoothness with different $C^\infty_B$-differentiabilities (see [2]). From [2] and [4] follow that a mapping between Fréchet spaces is smooth iff it is $C^\infty_B$-differentiable. Thus the inverse function theorem of Nash and Moser is valid for this concept of smoothness.

REFERENCES